Lecture 4

The set of equations reached at in the previous lecture are known as Lorentz transformation of space and time. Space and time have become inseparable, they are linked into something called spacetime. It is of course a consequence of the fact the speed of light, a space interval traveled in a time interval, is a universal constant. It possesses the same value in all inertial frames. The inverse transformation is obtained by replacing every v by –v and switch the prime, i.e. primed variables become unprimed and the primed ones are primed and S is considered to move with a velocity –v relative to S` which is considered to be stationary. The inverse transformation can then be expressed as:

$$x = \frac{x` - vt`}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(x` + vt`)$$

$$y = y`$$

$$z = z`$$

$$t = \frac{t` + vx`/c^2}{\sqrt{1 - \frac{v^2}{c}}} = \gamma(t` + vx`/c^2)$$

Let us now consider a case where v is much less than c . v<<<c:

$$\therefore A = K = 1$$

$$B = 0$$

which gives

$$x` = x - vt$$

$$y` = y$$

$$z` = z$$

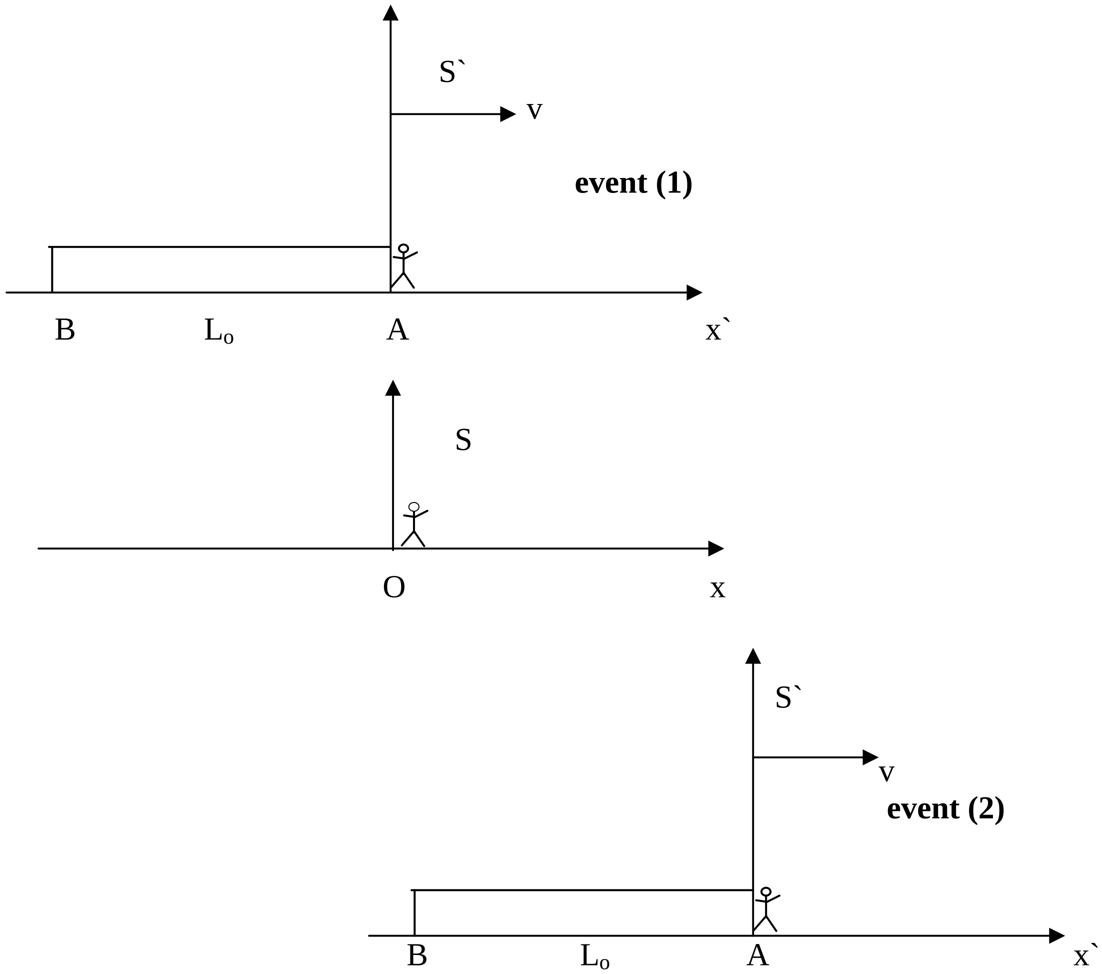
$$t` = t$$

Which takes us back to the Galilean or classical transformation.

4.1 Application to Lorentz Transformation

1. **Lorentz contraction or relativity of length**

Assume a length L_o is place along the x-axis in the inertial frame S`, determine the length as seen by the observer O. To do so, we will to think of a set up where two events are taken place and recorded by the two observers from which the length can be evaluated.



The question now is what the observer O measures the length L_o.

Event(1): A and O coincide

O sees $x_1 = 0$, $t_1 = 0$

O` sees $x`_1 = 0$, $t`_1 = 0$

Event (2): B and O coincide

O sees $x_2 = 0$, $t_2 = \Delta t = \frac{L}{v}$

O` sees $x`_2 = -L_o$, $t`_2 = \Delta t` = \frac{L_o}{v}$

Using Lorentz transformation

$$x`_1 = \gamma(x_1 - vt_1)$$

$$x`_2 = \gamma(x_2 - vt_2)$$

$$x`_2 - x`_1 = \gamma[(x_2 - x_1) - v(t_2 - t_1)]$$

$$-L_0 = -\gamma v \Delta t$$

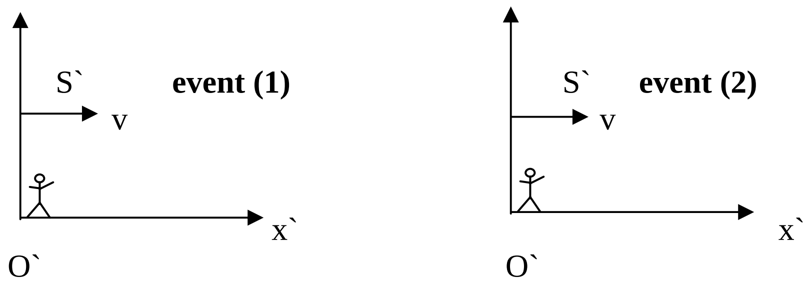
$$L_o = \gamma v \frac{L}{v} = \gamma L$$

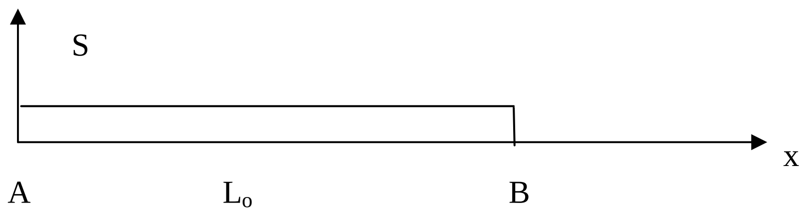
$$\therefore L = \frac{L_o}{\gamma} \qquad < L_o$$

i.e. the moving object gets contracted along the direction of motion. The proper length is the longer one.

Inverse Situation:

Suppose now we place L_o in S, the stationary frame and asked O` to measure it.





Event (1) : O` and A coincide

O sees : $x_1 = 0$, $t_1 = 0$

O` sees : $x`_1 = 0$, $t_1 = 0$

Event (2) : O` and B coincide

O sees : $x_2 = L_o$, $t_2 = \frac{L_o}{v}$

O` sees : $x`_2 = 0$, $t`_2 = \frac{L}{v}$

Using the inverse transformation:

$$x_1 = \gamma(x`_1 + vt`_1)$$

$$x_2 = \gamma(x`_2 + vt`_2)$$

$$x_2 - x_1 = \gamma[(x`_2 - x`_1) + v(t`_2 - t`_1)]$$

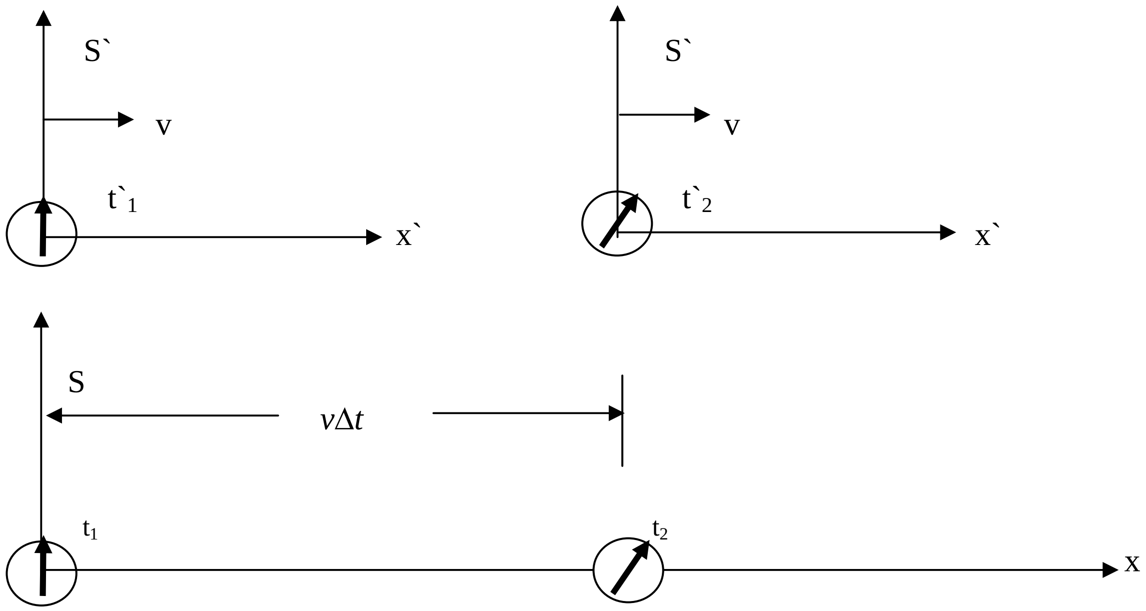
$$\therefore L_o = \gamma v \Delta t` = \gamma v \frac{L}{v} = \gamma L$$

$$again \quad L = \frac{L_o}{\gamma}$$

and the proper length is the longer one.

2. Time Dilation, or relativity of time

Let us examine the case where a time interval is being measured and recorded in S and S`



Event (1) : The two clocks coincide

O : records $x_1 = 0$, $t_1 = 0$

O` : records $x`_1 = 0$, $t`_1 = 0$

Event (2) : S` has traveled some distance

O : records $x_2 = v\Delta t$, $t_2 = \Delta t$

O` : records $x`_2 = 0$, $t`_2 = \Delta t`$

$\Delta t`$ is the time duration O` has recordered in his frame, it is a real time, while Δt is that duration between the two events as recordered by O. Using Lorentz transformation:

$$t`_1 = \gamma(t_1 - \frac{vx_1}{c^2})$$

$$t`_2 = \gamma(t_2 - \frac{vx_2}{c^2})$$

$$\therefore \Delta t` = t`_2 - t`_1 = \gamma[\Delta t - \frac{v}{c^2}(x_2 - x_1)]$$

$$but \;\; x_2 - x_1 = v\Delta t$$

$$\therefore \;\; \Delta t` = \gamma[\Delta t - \frac{v^2}{c^2}\Delta t]$$

$$= \gamma\Delta t[1 - \frac{v^2}{c^2}] = \Delta t\sqrt{1 - \frac{v^2}{c^2}}$$

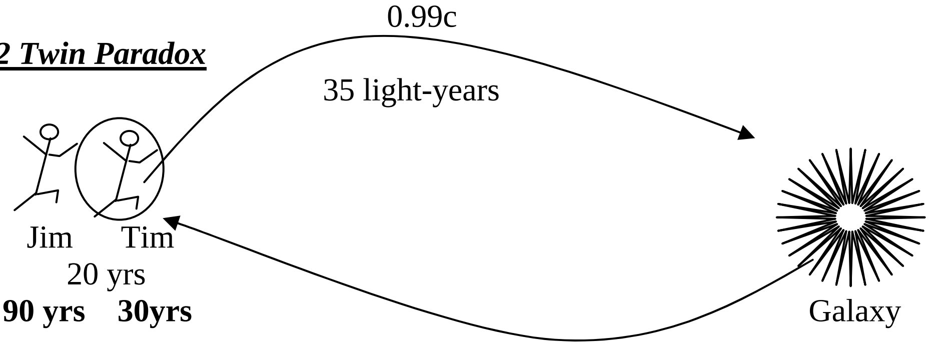
$$i.e. \quad \Delta t = \frac{\Delta t`}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\Delta t`$$

$$\therefore \qquad \Delta t > \Delta t`$$

The reverse is also true.

It is clear that the time has been dilated. The time duration between the two events is longer as recorded by O. This means that the clock in S` runs slower as seen by O`. The reverse is also true, an observer in S` would conclude that the clock in S is the one which runs slower. That is to say, each observer will see the other to have slower clock, and be aging more slowly. This phenomenon is called time dilation. It has been verified in recent years by flying very accurate clocks around the world on jetliners and finding they register less time, by the predicted amount, than identical clocks left on the ground. Time dilation is very easy to observe in elementary particle physics, as we shall discuss later. It should be emphasized that by “clock” we mean any system that measures time intervals, a digital watch, a normal heartbeats, or the half-life time of a radioactive isotopes.

4.2 Twin Paradox



$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-(0.99)^2}} \cong 7$$

Perhaps the most famous of the paradoxes of special relativity, which was still being hotly debated in national journals in the fifties, is the twin paradox. The scenario is as follows. One of two twins – Tim-- is an astronaut. He sets off in a relativistic spaceship to some Galaxy, 35 light-years away, at a speed of, say, 0.99c. When Tim gets there, he immediately turns around and comes back. As seen by his brother on earth, Tim’s clock ran slowly by the time dilation factor $\sqrt{1-\frac{v^2}{c^2}}$, so although the round trip took 70/0.99 years ~ 70years by earth time, Tim has only aged by 1/7 of that, or 10 years. So as Tim steps down out of the spaceship, he is 60 years younger than his twin brother, Jim. But wait a minute -- how does this look from Tim’s point of view? He sees the earth to be moving at 0.99c, first away from him then towards him. So Tim must see his brother's clock on earth to be running slow! So doesn't Tim expect his brother Jim on earth to be the younger one after this trip?

The key to this paradox is that this situation is not as symmetrical as it looks. The two twins have quite different experiences. The one on the spaceship is not in an inertial frame during the initial acceleration and the turnaround and braking periods. (To get an idea of the speeds involved, to get to 0.6c at the acceleration of a falling stone would take over six months.) Our analysis of how a clock in one inertial frame looks as viewed from another doesn't work during times when one of the frames isn't inertial - in other words, when one is accelerating.

4.3 Simultaneity and causality

Consider two event taking place simultaneously and independently in the S frame. The coordinates of the two events are:

$$(x_1, y_1, z_1, t_1)$$
$$(x_2, y_2, z_2, t_1)$$

While the S``'s observer would see the two events in his frame taking place at the following coordinates:

$$(x`_1, y`_1, z`_1, t`_1)$$
$$(x`_2, y`_2, z`_2, t`_2)$$

Using Lorentz transformation:

$$x`_1 = \gamma(x_1 - vt_1) \quad , \quad x`_2 = \gamma(x_2 - vt_2)$$
$$y`_1 = y_1 \quad , \quad y`_2 = y_2$$
$$z`_1 = z_1 \quad , \quad z`_2 = z_2$$
$$t`_1 = \gamma(t_1 - \frac{v}{c^2} x_1) \quad , \quad t`_2 = \gamma(t_2 - \frac{v}{c^2} x_2)$$
$$\therefore t`_2 - t`_1 = \gamma[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)]$$

But since the two events occur simultaneously in the s-frame , i. e. $t_2 - t_1 = 0$.

$$\therefore t`_2 - t`_1 = \frac{\gamma}{vc^2}(x_1 - x_2)$$

So, two simultaneous events in an inertial frame are not necessarily simultaneous in another inertial frame.