Control Systems And Their Components (EE391)

Lec. 11: Closed loop SS Control (State Estimators Continued)

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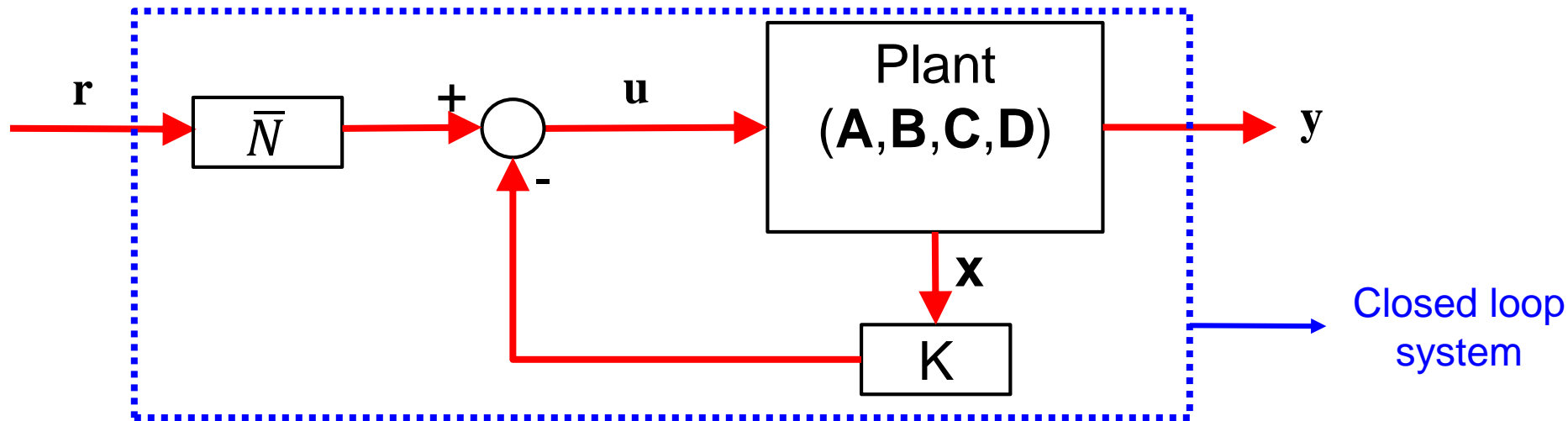
Lecture Outline

- ❑ Introduction to state estimators/observers
- ❑ Open loop vs Closed loop estimators
- ❑ Combining regulator with state estimator

Intro to State estimators / observers

- What we did so far

- Problem is that we have assumed **full state feedback** which means we have full access to the state variables of the system from which $u = \bar{N} r - \mathbf{K}\mathbf{x}$ is evaluated
- This is not true since in reality we only have access to the sensor outputs y and not the state variables **x**
- Could try output feedback but will have less degrees of freedom compared to state feedback (cannot control all pole locations freely like what we did with **K**)

Intro to State estimators / observers

- The solution to the lack of measurements of **x** is to use **<u>state estimator/observer</u>**
- A state estimator/observer is a replica of the actual system or plant that tries to estimate the true state variables of the system from the actual measured output y and provides the estimated state vector $\hat{\mathbf{x}}$
- We can then combine the developed estimator together with state feedback control to have a realistic method of controlling the closed loop poles based on the feedback of estimated state variables $u = -\mathbf{K}\hat{\mathbf{x}}$ (more on this later but we will focus on estimator alone for the moment)

- Estimation strategies we have in hand
  - ➢ Open loop (bad strategy as we will see)
  - ➢ Closed loop

Open loop estimator
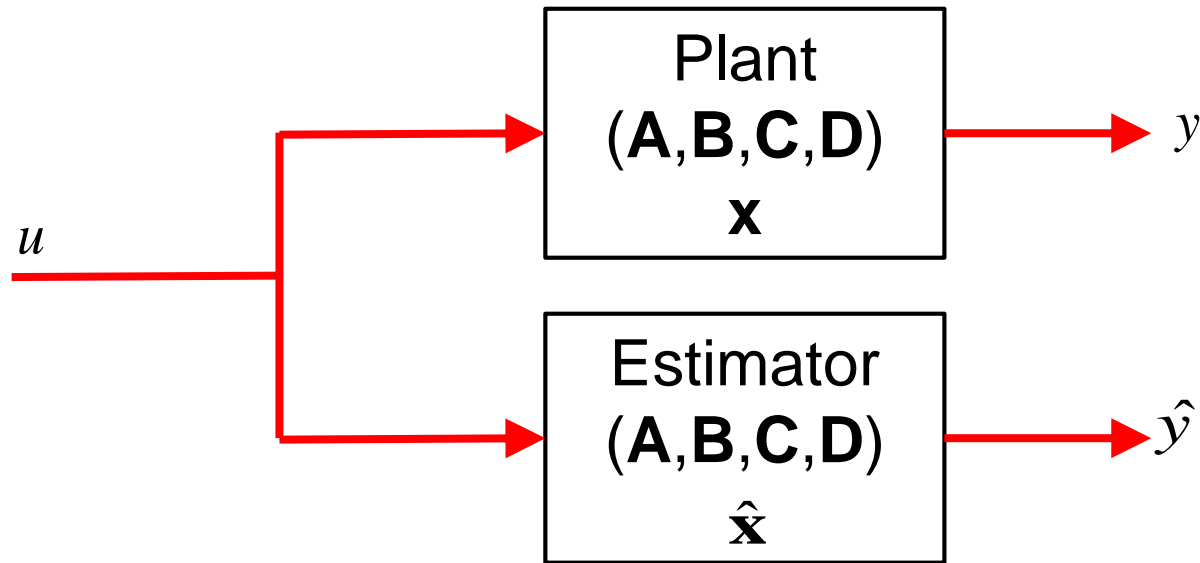
- Assuming we know the input u and plant matrixes A, B, C, and that D = 0
- We can just simulate a replica of the actual plant on say a computer and obtain an estimate $\hat{\mathbf{x}}$ as follows

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t)$$

Open loop estimator

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \quad \Longleftarrow \quad \boxed{\text{Dynamic eq. of actual plant}}$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) \quad \Longleftarrow \quad \boxed{\substack{\text{Dynamic eq. of simulated plant} \\ \text{(estimator)}}}$$

$$\text{If} \quad \mathbf{x}(0) = \hat{\mathbf{x}}(0), \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \quad \forall t$$

- However we do not know $\mathbf{x}(0)$ so how well the above estimator works if the initial estimation error is not zero
- Define the estimation error $\mathbf{e}(t)$

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

$$\frac{d}{dt}\{\mathbf{x}(t) - \hat{\mathbf{x}}(t)\} = \mathbf{A}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)$$

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t)$$

$$\therefore \boxed{\mathbf{e}(t) = e^{\mathbf{A}t}\mathbf{e}(0)}$$

Open loop estimator

$$\mathbf{e}(t) = e^{\mathbf{A}t}\mathbf{e}(0)$$

- Everything looks fine if initial error $\mathbf{e}(0) = 0$
- If $\mathbf{e}(0) \neq 0$, $\mathbf{e}(t)$ as $t \to \infty$ may decay to zero if the eigenvalues of $\mathbf{A}$ have negative real part (if the original plant is stable)
- Since the estimation error is totally dependent on $\mathbf{A}$, this is not a good estimation strategy since we cannot control the dynamics of the estimation error at all
- We may make use of other available information in building a better state estimator (how? → closed loop estimator)
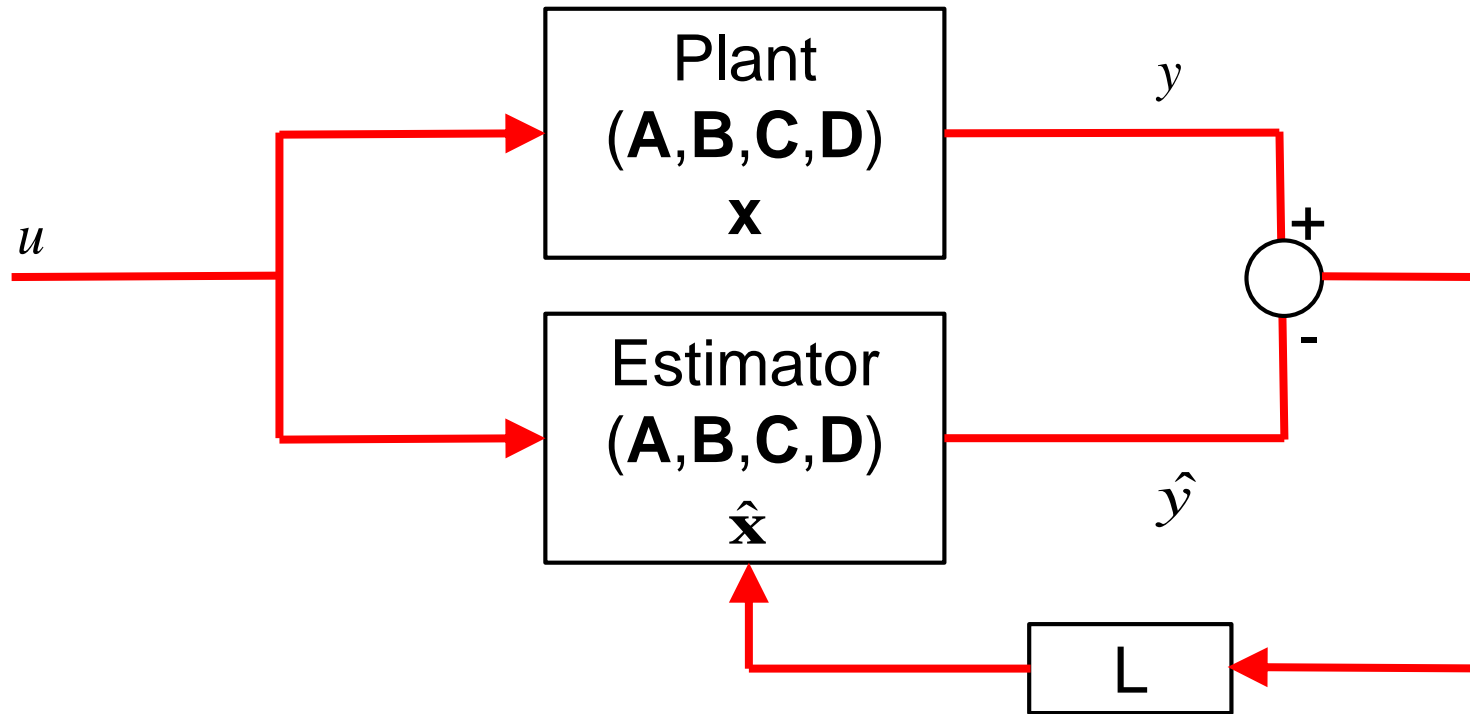
Closed loop estimator
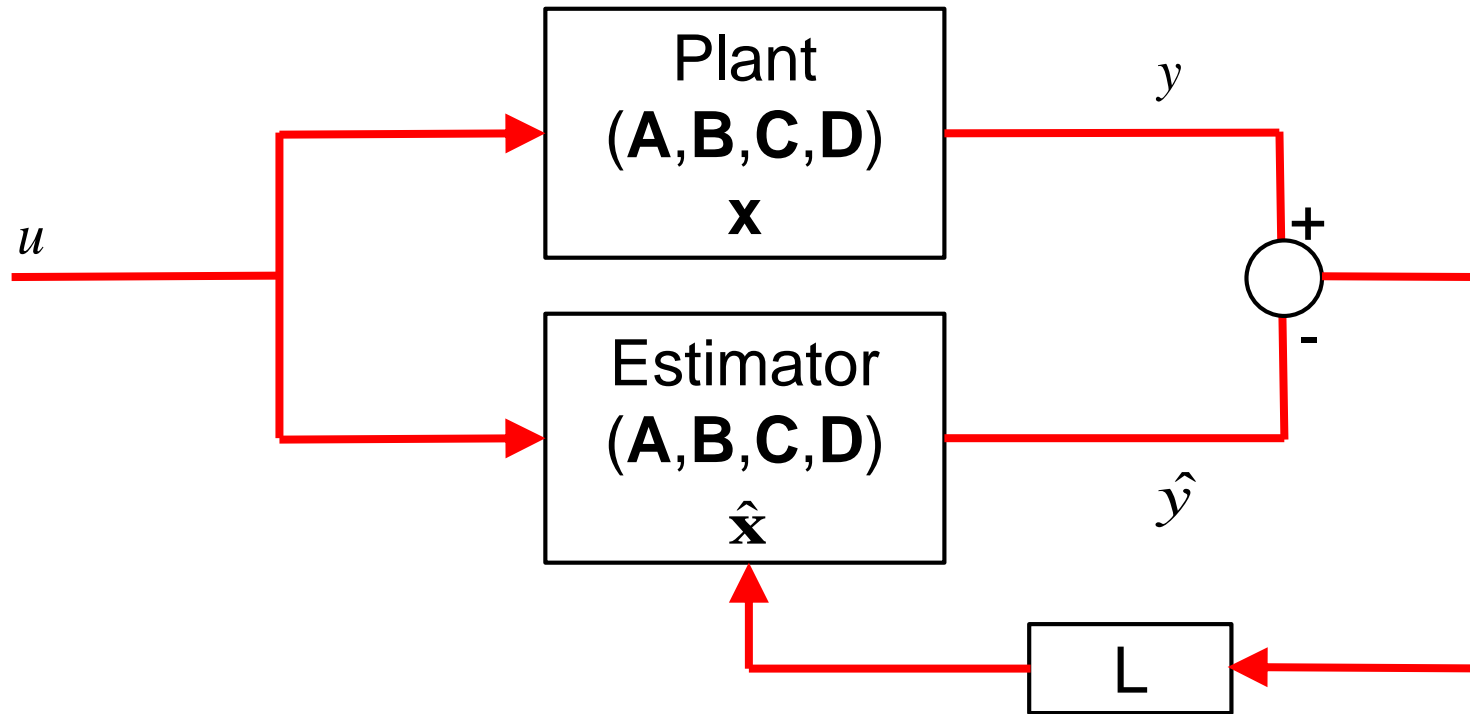
- The idea is to feedback the error in the estimated output, i.e. its difference from the actual output of the system which can be observed
- **L** is a selectable gain matrix (similar to **K**) that will allow us to control the dynamics of the estimation error **e**(t) as will be seen

Closed loop estimator

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

Dynamic eq. of actual plant

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(y(t) - \hat{y}(t)\right)$$

Dynamic eq. of simulated plant (estimator)

Closed loop estimator

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

Dynamic eq. of actual plant

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(y(t) - \hat{y}(t)\right)$$

Dynamic eq. of simulated plant (estimator)

$$y(t) = \mathbf{C}\mathbf{x}(t)$$

Output eq. of actual plant

$$\hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$$

Output eq. of estimator

- Let's try to find the dynamics of $\mathbf{e}(t)$ with the added feedback to the estimator

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) - \mathbf{L}\left(y(t) - \hat{y}(t)\right)$$

$$= \mathbf{A}\mathbf{e}(t) - \mathbf{L}\mathbf{C}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)$$

$$= \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\mathbf{e}(t)$$

Closed loop estimator

$$\dot{\mathbf{e}}(t) = \left(\mathbf{A} - \mathbf{LC}\right)\mathbf{e}(t)$$

$$\therefore \boxed{\mathbf{e}(t) = e^{\left(\mathbf{A} - \mathbf{LC}\right)t}\mathbf{e}(0)}$$

- It is obvious that by choosing a proper gain matrix **L**, we can control the dynamics of the estimation error, i.e. make it go to zero fast such that the estimated state variables converge to the actual state variables fast enough
- This is all controlled by the eigenvalues of **A-LC**

$$\left| s\mathbf{I} - \left(\mathbf{A} - \mathbf{LC}\right) \right| = \prod_{j=1}^{n} \left(s - s_j \right) = 0$$

Desired pole locations of state estimator where? → we will see later

Controller and Observer design (Dual problems)

Controller design

$$\mathbf{K} \in \mathbf{R}^{1 \times n}$$

$$\left| s\mathbf{I} - (\mathbf{A} - \mathbf{BK}) \right| = \prod_{j=1}^{n} (s - s_j) = 0$$

Desired pole locations closed loop sys

Observer/Estimator design

$$\mathbf{L} \in \mathbf{R}^{n \times 1}$$

$$\left| s\mathbf{I} - (\mathbf{A} - \mathbf{LC}) \right| = \prod_{m=1}^{n} (s - s_m) = 0$$

Desired pole locations of state estimator

- **K** and **L** are chosen to achieve desired pole locations
- **Controller** and **Observer** design are called **dual problems**
- Just like before when the system had to be **controllable** to find **K**, the system now has to be **observable** to find **L**

Akermann's formula for Observer design

- It gives a formal way to obtain **L**
- Without proof

$$\mathbf{L} = \mathbf{O}_n^{-1}\mathbf{\Phi}_{desired}(\mathbf{A})\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^{\mathbf{T}}$$

Observability matrix

Desired characteristic equation of state estimator/observer

- Clearly **O**$_n$ needs to be invertible, hence full rank, hence the system must be **observable** in order to be able to find **L** to place the estimator poles freely

Observer design

Example For the system with the following matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Test observability
- Find **L** that makes poles of estimator/observer at -3 and -4

Observer design

Solution For the system with the following matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Test observability

$$\mathbf{O}_n = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1.5 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\{\mathbf{O}_n\} = 2 \quad \Rightarrow \quad \text{observable}$$

Observer design

Solution For the system with the following matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Find **L** that makes poles of estimator/observer at -3 and -4

$$\left| s\mathbf{I} - (\mathbf{A} - \mathbf{LC}) \right| = \prod_{m=1}^{n} (s - s_m) = 0$$

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right| = (s+3)(s+4)$$

$$\left| \begin{bmatrix} s+1+L_1 & -1.5 \\ L_2 - 1 & s+2 \end{bmatrix} \right| = (s+3)(s+4)$$

Observer design

Solution For the system with the following matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Find **L** that makes poles of estimator/observer at -3 and -4

$$\left| \begin{bmatrix} s+1+L_1 & -1.5 \\ L_2-1 & s+2 \end{bmatrix} \right| = (s+3)(s+4)$$

$$s^2 + (L_1+3)s + (2L_1+1.5L_2+0.5) = s^2 + 7s + 12$$

$$L_1 = 4$$

$$L_2 = 2.333$$

MATLAB
Use $\mathbf{A}^T$ and $\mathbf{C}^T$ as your **A** and **B** in "place" function
L = place(A.',C.',desired poles)

Augmented SS equations

- It is common to augment the estimation error to the original true state vector to form the following augmented SS equations representing the dynamic behavior of **both the Plant and Estimator** to facilitate simulating both the plant and estimator together

- **For a closed loop state estimator**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\dot{\mathbf{e}}(t) = \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\mathbf{e}(t)$$

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}$$

- It is common to augment the estimation error to the original true state vector to form the following augmented SS equations representing the dynamic behavior of **both the Plant and Estimator** to facilitate simulating both the plant and estimator together

- **For an open loop state estimator**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t)$$

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}$$

Comparison between closed and open loop estimator

Example For the system with the following matrices and initial state vector as well as initial estimated state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0, \quad \mathbf{x}(0) = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}, \quad \hat{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Use MATLAB to show the state variables versus time together with the estimated state variables assuming both an **open loop** and a **closed loop estimator** (use an input signal and closed loop estimator poles of your preference)
- Plot the estimation error in both cases
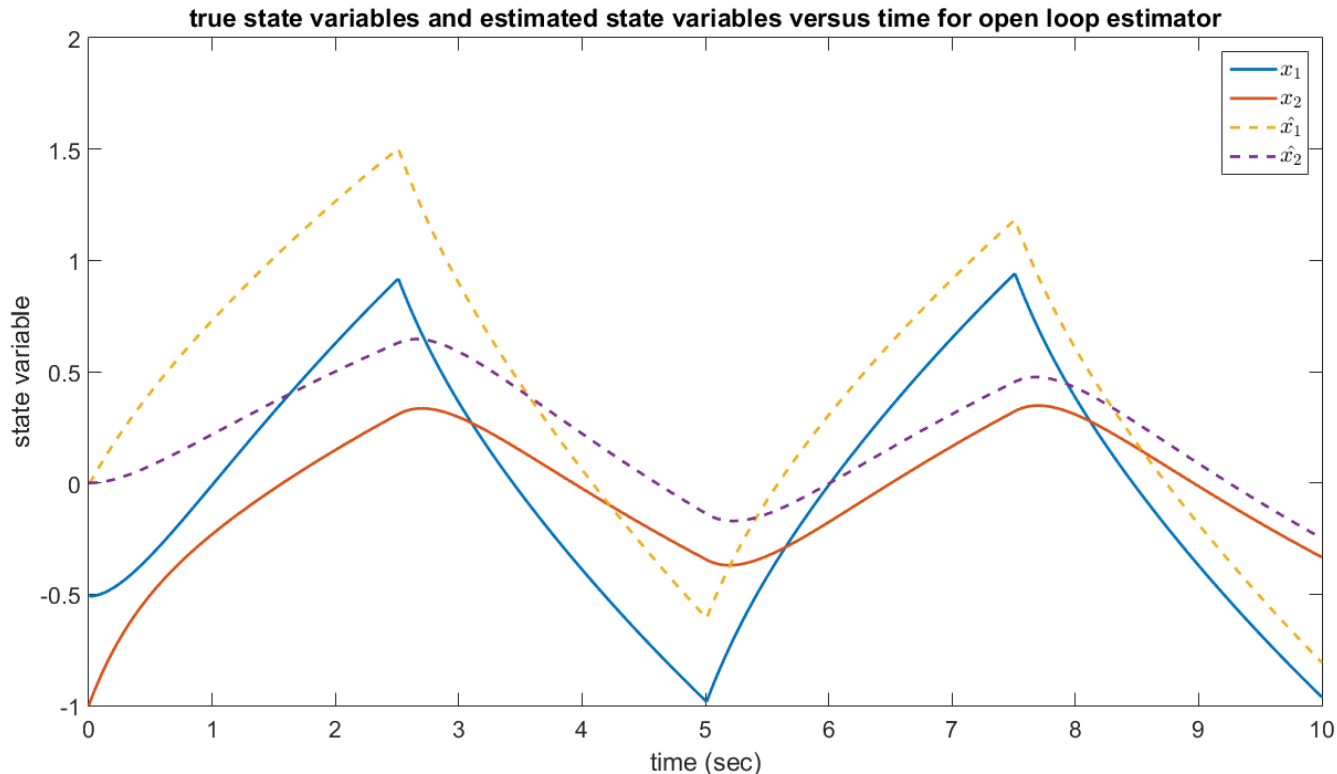- Comment on the convergence of both estimators

Comparison between closed and open loop estimator

Example For the system with the following matrices and initial
state vector as well as initial estimated state vector

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true state variables and estimated state variables versus time for open loop estimator

Example For the system with the following matrices and initial state vector as well as initial estimated state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0, \quad \mathbf{x}(0) = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}, \quad \hat{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
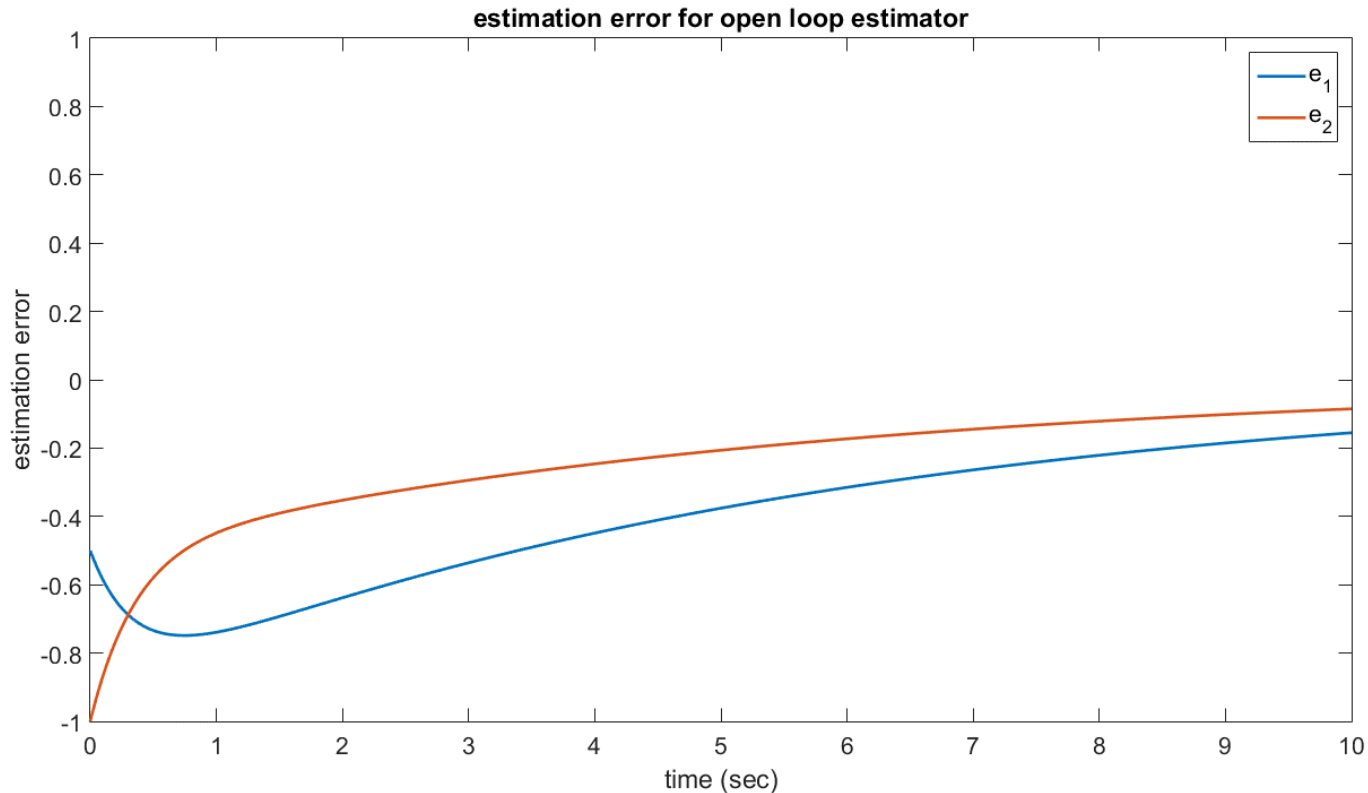

estimation error for open loop estimator

Comparison between closed and open loop estimator

Example For the system with the following matrices and initial
state vector as well as initial estimated state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0, \quad \mathbf{x}(0) = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}, \quad \hat{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
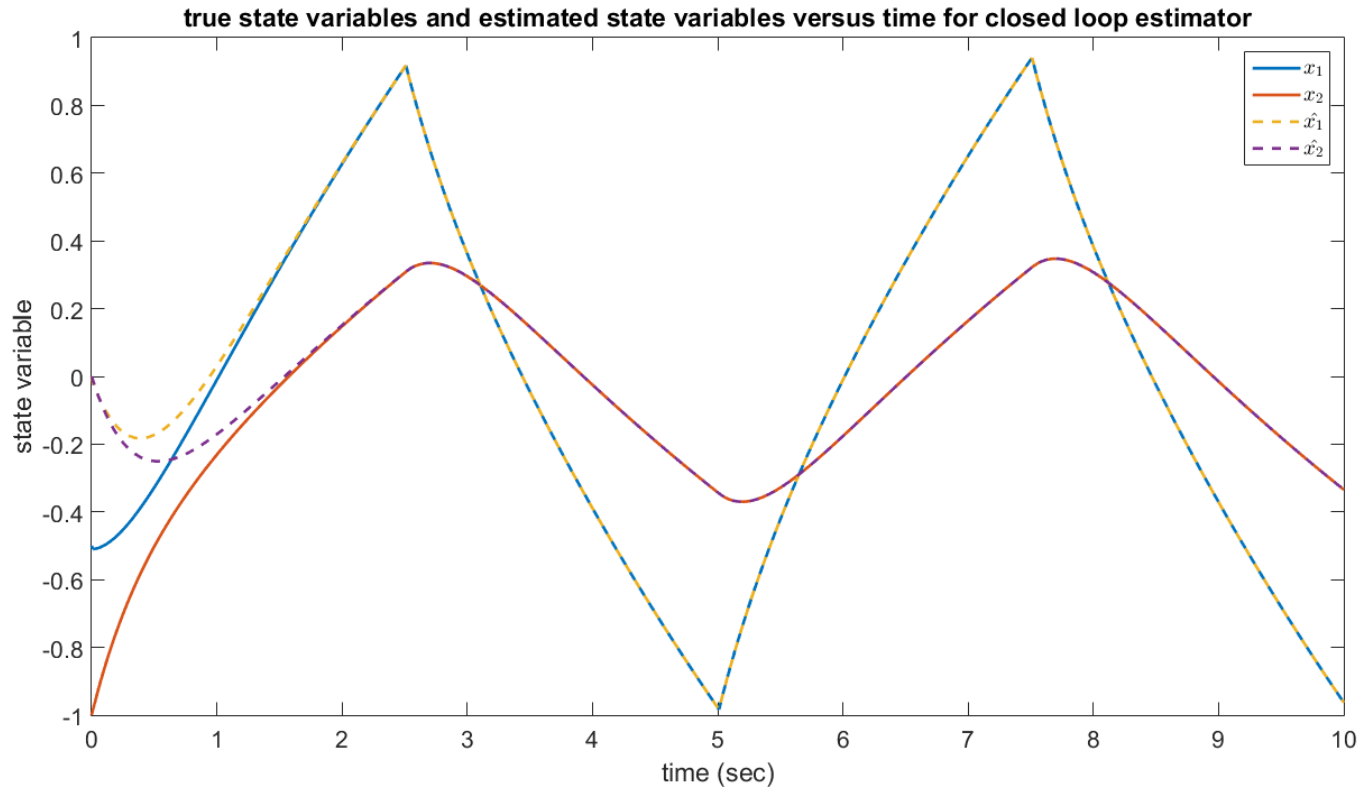

true state variables and estimated state variables versus time for closed loop estimator

Comparison between closed and open loop estimator

Example For the system with the following matrices and initial state vector as well as initial estimated state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0, \quad \mathbf{x}(0) = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}, \quad \hat{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
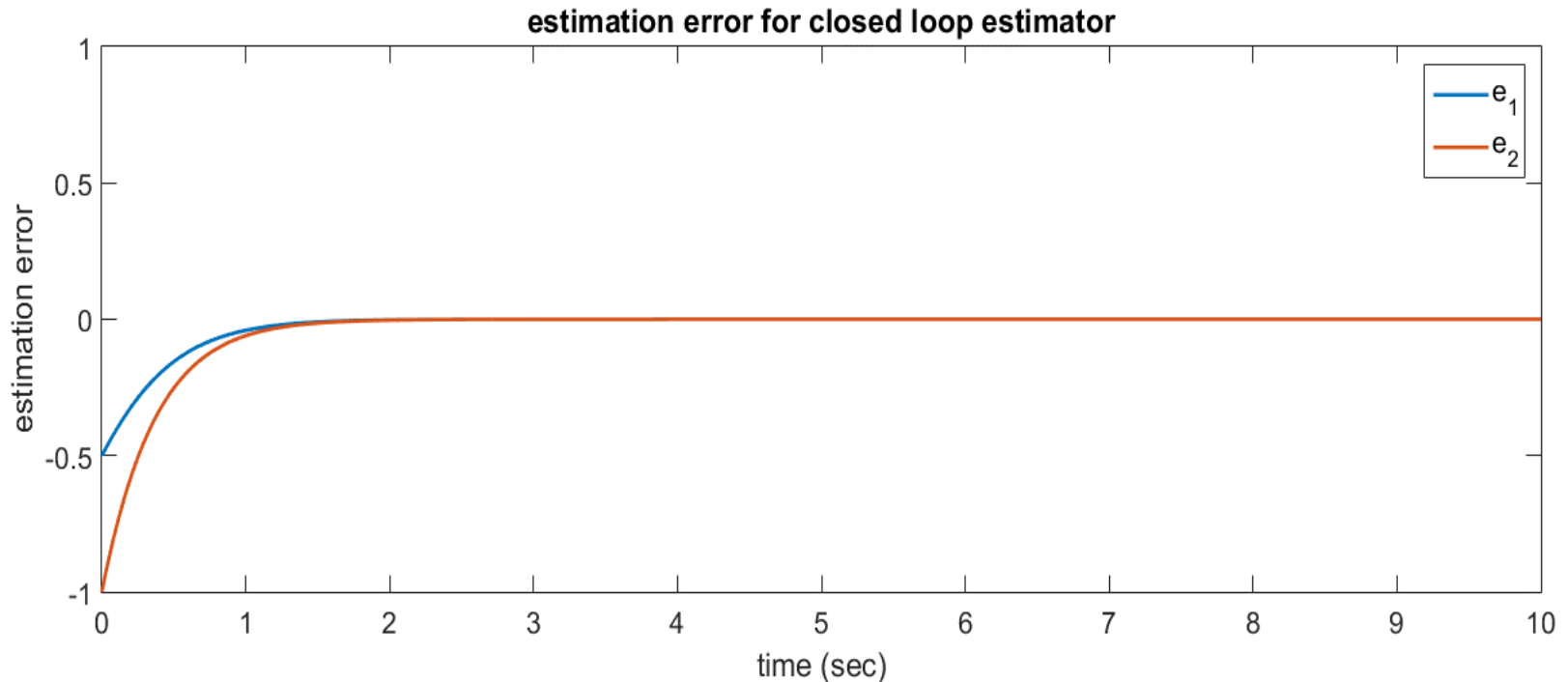

estimation error for closed loop estimator

Combining regulator with state estimator

- Now we will study the performance when combining a regulator **designed as** $u = -\mathbf{K}\mathbf{x}$ **but implemented as** $u = -\mathbf{K}\hat{\mathbf{x}}$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \qquad \Leftarrow \text{ plant state equation}$$

$$= \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t) \qquad \Leftarrow \text{ apply state feedback } u = -\mathbf{K}\hat{\mathbf{x}}$$

$$= \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\left(\mathbf{x}(t) - \mathbf{e}(t)\right)$$

$$= \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{e}(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\mathbf{C}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)$$

$$\dot{\mathbf{e}}(t) = \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\mathbf{e}(t)$$

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ 0 & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} \qquad \Leftarrow \text{ augmented equation}$$

$$y(t) = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}$$

Combining regulator with state estimator

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ 0 & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} \qquad \Leftarrow \quad \text{augmented equation}$$

$$\mathbf{A}_{cl}$$

- This says that the dynamics of both the state vector of the closed loop systems after introducing feedback as well as the estimation error are determined by the eigenvalues of $\mathbf{A}_{cl}$
- Since this is a block upper diagonal matrix, its eigenvalues are given by

$$\left| s\mathbf{I} - \mathbf{A}_{cl} \right| = \left| s\mathbf{I} - \left(\mathbf{A} - \mathbf{BK} \right) \right| \cdot \left| s\mathbf{I} - \left(\mathbf{A} - \mathbf{LC} \right) \right|$$

- This means that **the poles of the closed loop system are the union of the regulator and estimator poles**

Combining regulator with state estimator

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ 0 & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} \qquad \Leftarrow \quad \text{augmented equation}$$

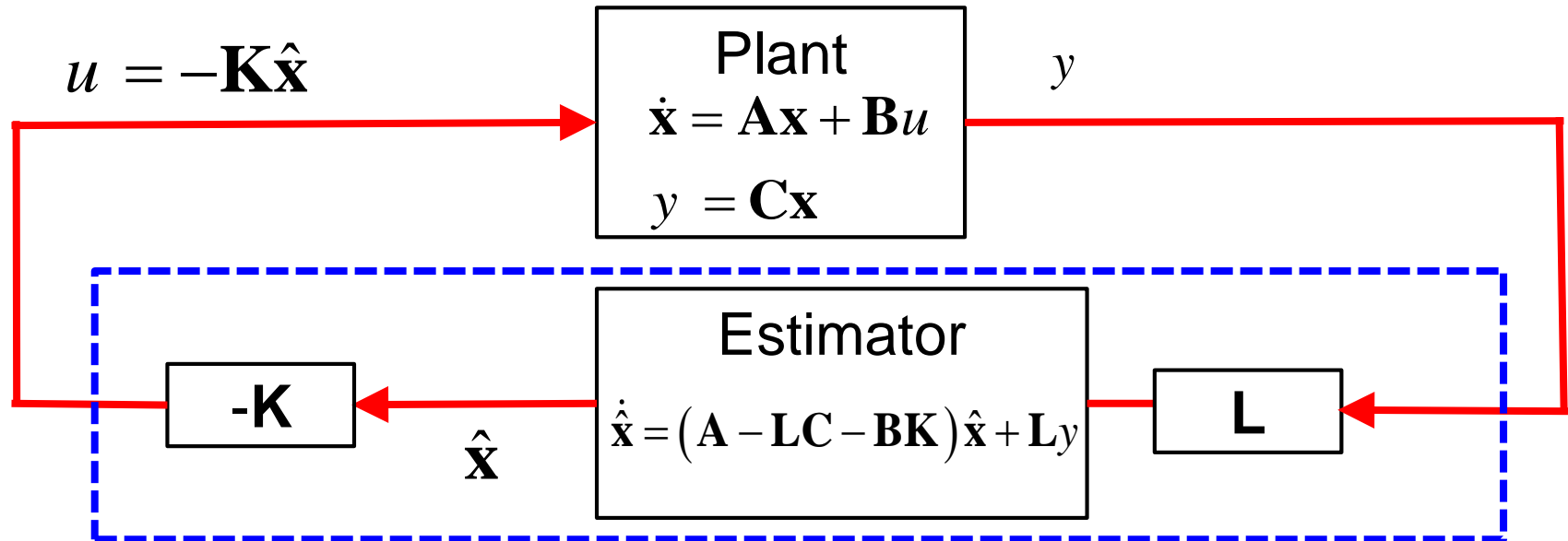$$\underbrace{\qquad\qquad\qquad\qquad}_{\mathbf{A}_{cl}}$$

$$\left| s\mathbf{I} - \mathbf{A}_{cl} \right| = \left| s\mathbf{I} - \left(\mathbf{A} - \mathbf{BK} \right) \right| \cdot \left| s\mathbf{I} - \left(\mathbf{A} - \mathbf{LC} \right) \right|$$

- This also means that you can design the compensator and estimator separately just as we did before and then combine them **(separation principle)**
- As a design rule, you should place the estimator poles at >2 the real part of the regulator poles (found from transient specs required) to ensure the estimator converges fast and hence the estimated values used for feedback are good

Combining regulator with state estimator

- The whole system (regulator + estimator) looks like

$$u = -\mathbf{K}\hat{\mathbf{x}}$$

Plant
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

y

-K

$\hat{\mathbf{x}}$

Estimator
$$\dot{\hat{\mathbf{x}}} = \left(\mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K}\right)\hat{\mathbf{x}} + \mathbf{L}y$$
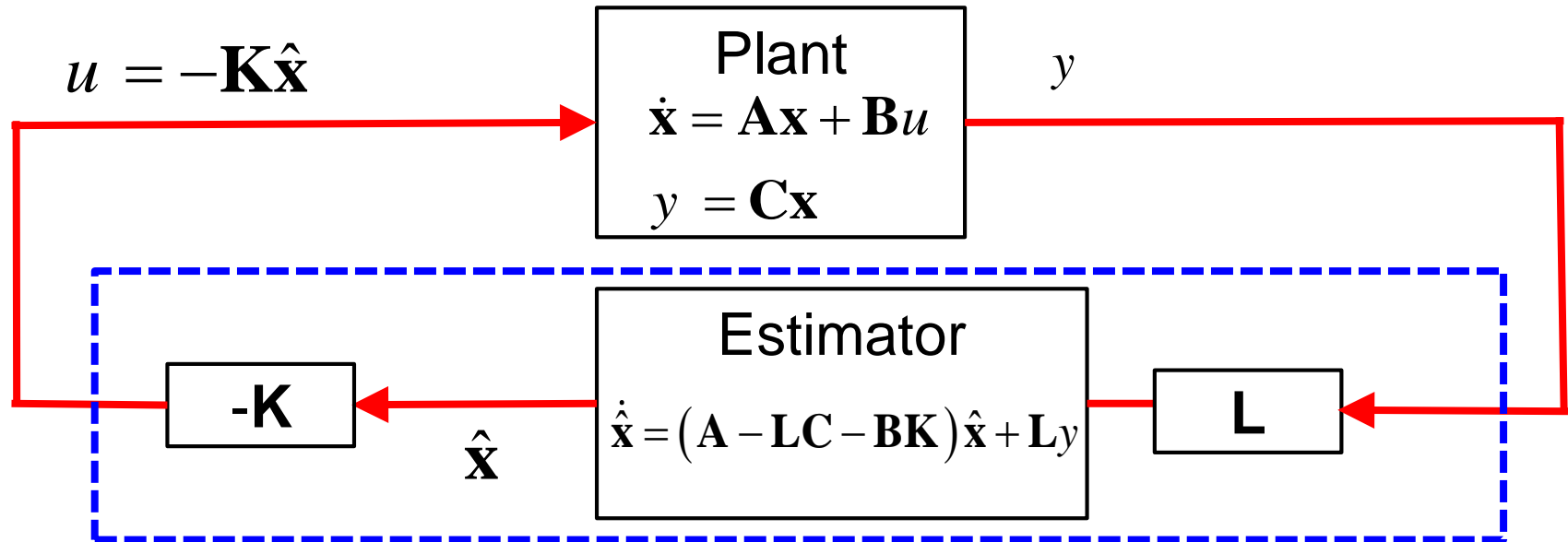
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The full compensator
(Regulator + Estimator)

- Compensator accepts the sensor outputs as its inputs and provides at its output the actuator input

Full compensator equations

$$u = -\mathbf{K}\hat{\mathbf{x}}$$

Plant
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

y

-K

Estimator
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K})\hat{\mathbf{x}} + \mathbf{L}y$$

$\hat{\mathbf{x}}$

L

The full compensator
(Regulator + Estimator)

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K})\hat{\mathbf{x}} + \mathbf{L}y$$

$$u = -\mathbf{K}\hat{\mathbf{x}}$$

SS Matrices of compensator

$$\mathbf{A}_c \equiv \mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K}, \quad \mathbf{B}_c \equiv \mathbf{L}, \quad \mathbf{C}_c \equiv -\mathbf{K}$$

$$\text{TF of compensator} = -\mathbf{K}\left(s\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K})\right)^{-1}\mathbf{L}$$