Control Systems And Their Components (EE391)

Lec. 2: Transfer Functions & Block Diagrams

Sat. Feb. 20th, 2016

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Lecture Outline

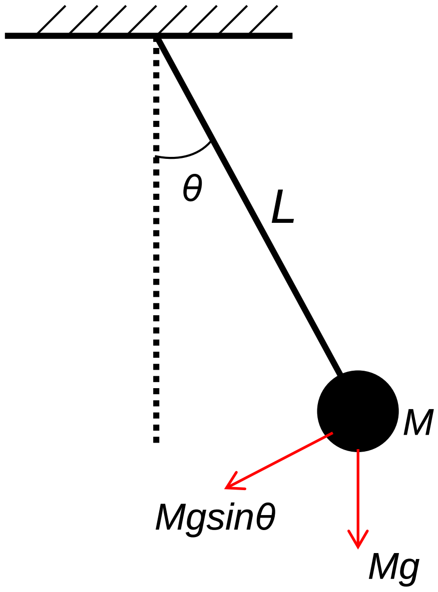
- Linearization of Nonlinear Systems
- Laplace Transform and Solution of Linear Differential Equations
- Transfer Functions of LTI Systems
- Block Diagram Representations

Linearization of nonlinear system

- Example: Pendulum oscillator

$$\sum Torques = J\ddot{\theta}$$

$$MgLsin\theta = J\ddot{\theta}$$
$$MgLsin\theta = ML^2\ddot{\theta}$$



$$ML\ddot{\theta} - Mgsin\theta = 0$$

$$\ddot{\theta} - \frac{g}{L}sin\theta = 0$$ ⇦ Nonlinear

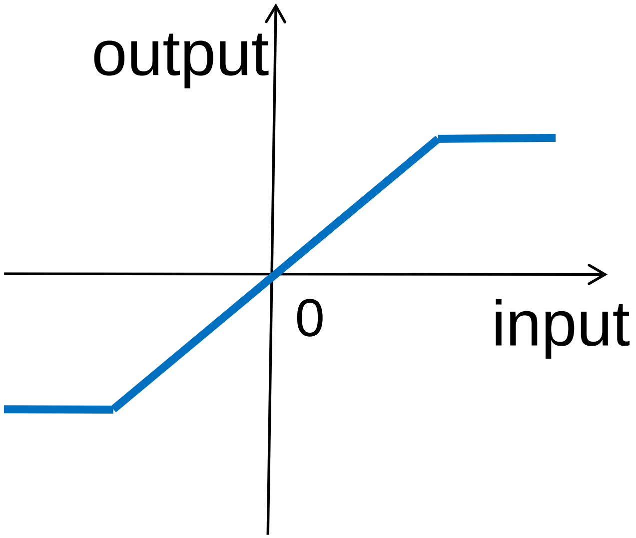
But for small θ $\quad sin\theta \approx \theta$

$$\ddot{\theta} - \frac{g}{L}\theta = 0$$ ⇦ After linearization

What is the formal way that we can use to linearize any model around the equilibrium point ??? Taylor series

Linearization of nonlinear system

Example of typical nonlinear characteristics in control system.



Saturation (Amplifier)

Linearization of nonlinear system

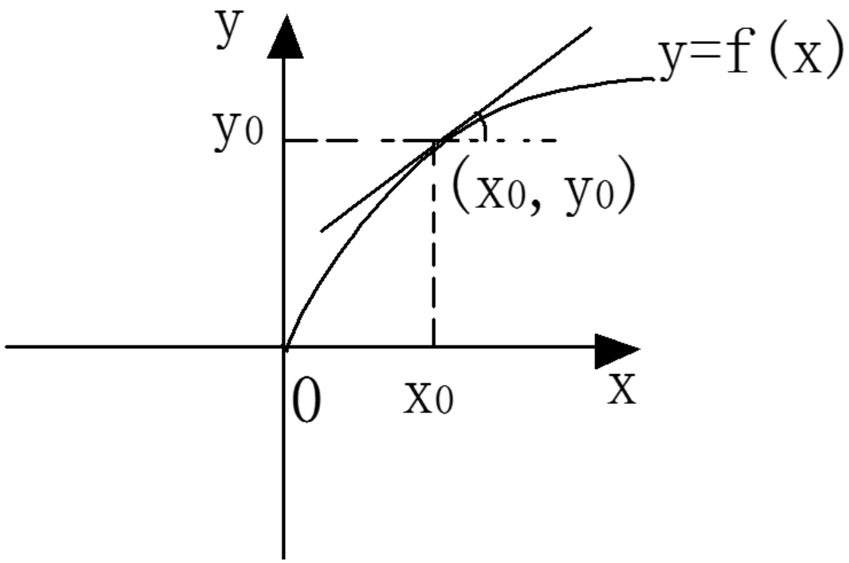
- Method of linearization
  - Assume the system is operating around an equilibrium / operating point
  - Represent the input and output by their values at the operating point plus a small perturbation or error
  - Expand the nonlinear i/o relationship using Taylor series around this equilibrium point and neglect all terms after the linear (first derivative term)
  - This is a very reasonable / practical way to use for linearization as long as the perturbation stays small enough around the equilibrium point

Linearization of nonlinear system

- Assume y=f(x) where f is a nonlinear function
- Assume (x_0,y_0) is the equilibrium point. Expanding the nonlinear function y=f(x) into a Taylor series about $x = x_0$ yields

$$y = f(x) = y_0 + \left.\frac{dy}{dx}\right|_{x_0} (x - x_0) + \frac{1}{2!}\left.\frac{d^2 y}{dx^2}\right|_{x_0} (x - x_0)^2 + \cdots\cdots$$

$$\approx f(x_0) + \left.\frac{dy}{dx}\right|_{x_0} (x - x_0)$$



Linearization of nonlinear system

- If the output is a nonlinear function of multiple variables $x_1, x_2, x_3, \dots x_n$
- Assume $\left(x_{1_0}, x_{2_0}, x_{3_0}, \dots x_{n_0}\right)$ is the equilibrium point. Expanding the nonlinear function $y = f(x_1, x_2, x_3, \dots x_n)$ into a Taylor series about $(x_{1_0}, x_{2_0}, x_{3_0}, \dots x_{n_0})$ yields

$$
\begin{aligned}
y &= f(x_1, x_2, x_3, \dots x_n) \\
&\approx f\left(x_{1_0}, x_{2_0}, x_{3_0}, \dots x_{n_0}\right) + \left.\frac{\partial f}{\partial x_1}\right|_{x_{1_0}, x_{2_0}, x_{3_0}, \dots x_{n_0}} \left(x_1 - x_{1_0}\right) \\
&+ \left.\frac{\partial f}{\partial x_2}\right|_{x_{1_0}, x_{2_0}, x_{3_0}, \dots x_{n_0}} \left(x_2 - x_{2_0}\right) + \left.\frac{\partial f}{\partial x_3}\right|_{x_{1_0}, x_{2_0}, x_{3_0}, \dots x_{n_0}} \left(x_3 - x_{3_0}\right) \\
&+ \cdots + \left.\frac{\partial f}{\partial x_n}\right|_{x_{1_0}, x_{2_0}, x_{3_0}, \dots x_{n_0}} \left(x_n - x_{n_0}\right)
\end{aligned}
$$

Linearization of NL Systems

- Example: Linearize the NL equation Z = XY in the regions 5 ≤ X ≤ 7, 10 ≤ Y ≤ 12. Find the error if the linearized equation is used to calculate Z when X = 5, Y = 10

Solution:

Choose equilibrium point as X_0 = 6 and Y_0 = 11   (mean of both ranges…why??)

Expand using Taylor series

$$Z = X_0 Y_0 + \left.\frac{df}{dX}\right|_{X_0, Y_0} (X - X_0) + \left.\frac{df}{dY}\right|_{X_0, Y_0} (Y - Y_0)$$

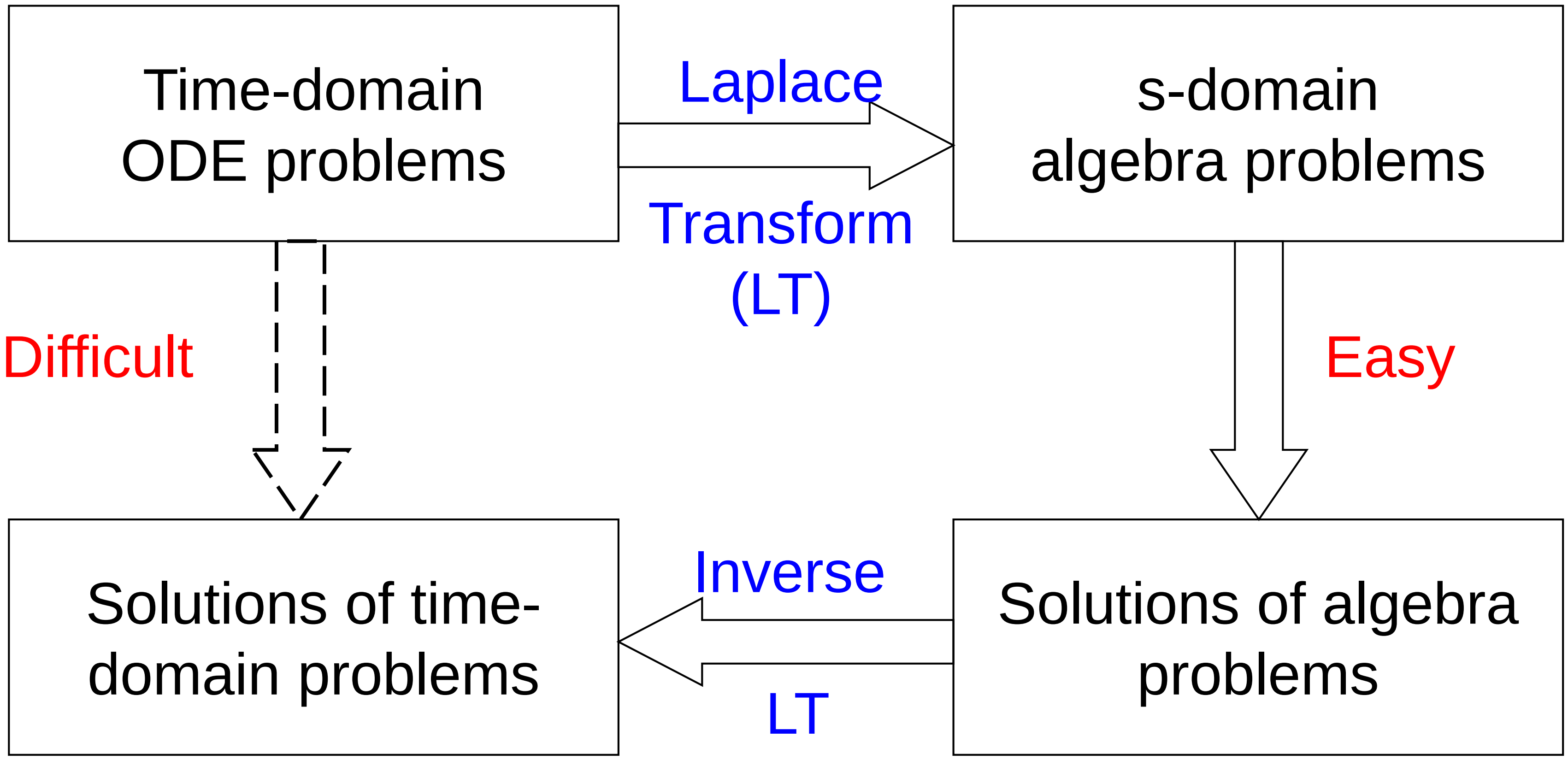
$$= 66 + 11(X - 6) + 6(Y - 11)$$

At X = 5 and Y = 10,

$$Z = 66 + 11(5 - 6) + 6(10 - 11) = 49$$

$$error = 49 - 5 \times 10 = -1$$

Laplace Transform for Solving Diff. Eq.



Laplace Transform for Solving Diff. Eq.

The Laplace transform of a function *f*(*t*) is defined as

$$F(s) = \mathcal{L}\left[f(t) \right]$$

$$= \int_0^\infty f(t) e^{-st} dt$$

where $s = \sigma + j\omega$ is a complex variable.

Laplace Transform for Solving Diff. Eq.

Examples

- Step signal: $f(t) = A$

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} A e^{-st} dt = -\frac{A}{s} e^{-st} \Big|_0^{\infty} = \frac{A}{s}$$

- Exponential signal $f(t) = e^{-at}$

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt \ = -\frac{1}{s+a} e^{-(a+s)t} \Big|_0^{\infty} = \frac{1}{s+a}$$

Laplace Transform for Solving Diff. Eq.

Laplace Transform Pairs of Common Signals

f(t)	F(s)	f(t)	F(s)
δ (t)	1	$\sin wt$	$\frac{w}{s^2 + w^2}$
1(t)	$\frac{1}{s}$	$\cos wt$	$\frac{s}{s^2 + w^2}$
t	$\frac{1}{s^2}$	$e^{-at} \sin wt$	$\frac{w}{(s+a)^2 + w^2}$
e^{-at}	$\frac{1}{s+a}$	$e^{-at} \cos wt$	$\frac{s+a}{(s+a)^2 + w^2}$

Laplace Transform for Solving Diff. Eq.

- Properties of Laplace Transform

(1) Linearity

$$\mathcal{L}[af_1(t)+bf_2(t)] = a\mathcal{L}[f_1(t)]+b\mathcal{L}[f_2(t)]$$

(2) Differentiation

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

Try to prove it !!

where f(0) is the initial value of f(t).

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

Laplace Transform for Solving Diff. Eq.

- Properties of Laplace Transform

(3) Integration

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

(4) Final-value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The final-value theorem relates the steady-state behavior of f(t) to the behavior of sF(s) in the neighborhood of s=0

(5) Initial-value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Laplace Transform for Solving Diff. Eq.

- <u>Properties of Laplace Transform</u>

(6) Shifting Theorem：

a. shift in time (real domain)

$$\mathcal{L}[f(t-\tau)] = e^{-\tau \cdot s} F(s)$$

b. shift in complex domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

(7) Real convolution (Complex multiplication)

$$\mathcal{L}\left[\int_0^t f_1(t-\tau) f_2(\tau) d\tau\right] = F_1(s) \cdot F_2(s)$$

Laplace Transform for Solving Diff. Eq.

- <u>Inverse Transform</u>

Inverse Laplace transform, denoted by $\mathcal{L}^{-1}[F(s)]$ is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi \cdot j} \int_{C-j\infty}^{C+j\infty} F(s) e^{st} ds \quad (t > 0)$$

where C is a real constant。

<u>Note</u>: The inverse Laplace transform operation involving rational functions can be carried out using Laplace transform table and partial-fraction expansion.

Laplace Transform for Solving Diff. Eq.

Partial-Fraction Expansion method for finding Inverse Laplace Transform

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad (m < n)$$

If F(s) is broken up into components

$$F(s) = F_1(s) + F_2(s) + \ldots + F_n(s)$$

If the inverse Laplace transforms of components are readily available, then

$$\mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[F_1(s)\right] + \mathcal{L}^{-1}\left[F_2(s)\right] + \ldots + \mathcal{L}^{-1}\left[F_n(s)\right]$$

$$= f_1(t) + f_2(t) + \ldots + f_n(t)$$

Laplace Transform for Solving Diff. Eq.

<u>Poles</u>

A complex number s_0 is said to be a pole of a complex variable function $F(s)$ if $F(s_0)=\infty$

<u>Zeros</u>

A complex number s_0 is said to be a zero of a complex variable function $F(s)$ if $F(s_0)=0$

Examples:

$$\frac{(s-1)(s+2)}{(s+3)(s+4)}$$ poles: -3, -4; zeros: 1, -2

$$\frac{s+1}{s^2+2s+2}$$ poles: -1+j, -1-j; zeros: -1

Laplace Transform for Solving Diff. Eq.

Case 1: F(s) has simple real poles

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Partial-Fraction Expansion

$$=\frac{c_1}{s-p_1}+\frac{c_2}{s-p_2}+\cdots+\frac{c_n}{s-p_n}$$

where $p_i (i=1,2,\cdots,n)$ are roots of $D(s)=0$, and

$$c_i=\left[\frac{N(s)}{D(s)}(s-p_i)\right]\Bigg|_{s=p_i}$$

Inverse LT

$$f(t)= c_1 e^{-p_1 t} + c_2 e^{-p_2 t} + \ldots + c_n e^{-p_n t}$$

Solution is a sum of exponentials with different magnitudes and exponents

Laplace Transform for Solving Diff. Eq.

Example:

$$F(s)=\frac{1}{(s+1)(s-2)(s+3)}=\frac{c_1}{s+1}+\frac{c_2}{s-2}+\frac{c_3}{s+3}$$

$$c_1=\left[\frac{1}{(s+1)(s-2)(s+3)}\cdot(s+1)\right]\Bigg|_{s=-1}=-\frac{1}{6}$$

$$c_2=\left[\frac{1}{(s+1)(s-2)(s+3)}\cdot(s-2)\right]\Bigg|_{s=2}=\frac{1}{15}$$

$$c_3=\left[\frac{1}{(s+1)(s-2)(s+3)}\cdot(s+3)\right]\Bigg|_{s=-3}=\frac{1}{10}$$

$$\therefore F(s)=-\frac{1}{6}\cdot\frac{1}{s+1}+\frac{1}{15}\cdot\frac{1}{s-2}+\frac{1}{10}\cdot\frac{1}{s+3}$$

$$\therefore f(t)=-\frac{1}{6}e^{-t}+\frac{1}{15}e^{2t}+\frac{1}{10}e^{-3t}$$

Laplace Transform for Solving Diff. Eq.

Case 2: F(s) has complex conjugate poles

Example: $\ddot{y}(t)+4\dot{y}(t)+5y(t)=0, y(0)=\dot{y}(0)=1$

$$s^2Y(s)-sy(0)-\dot{y}(0)+4sY(s)-4y(0)+5Y(s)=0$$

$$\left(s^2+4s+5\right)Y(s)=s+5$$

$$Y(s)=\frac{s+5}{s^2+4s+5}=\frac{A}{s-(-2+j1)}+\frac{B}{s-(-2-j1)}$$

$$A=0.5-j1.5 \quad \text{and} \quad B=0.5+j1.5$$

$$y(t)=\left(0.5-j1.5\right)e^{(-2+j)t}+\left(0.5+j1.5\right)e^{(-2-j)t}$$

$$=e^{-2t}\cos t+3e^{-2t}\sin t$$

Try MATLAB functions:
roots(D)
[r,p,k]=residue(N,D)

Laplace Transform for Solving Diff. Eq.

Case 3: F(s) has multiple order poles

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_{n-r})(s-p_i)^l}$$

$$=\frac{c_1}{s-p_1}+\cdots+\frac{c_{n-l}}{s-p_{n-l}}+\frac{b_l}{(s-p_i)^l}+\frac{b_{l-1}}{(s-p_i)^{l-1}}+\cdots+\frac{b_1}{s-p_i}$$

Simple poles — Multi-order poles

The coefficients corresponding to simple poles are determined as before

The coefficients corresponding to the multi-order poles are determined as follows

$$b_l=\left[F(s)\cdot(s-p_i)^l\right]\Big|_{s=p_1},\ b_{l-1}=\left\{\frac{d}{ds}\left[F(s)\cdot(s-p_i)^l\right]\right\}\Big|_{s=pi},\cdots,$$

$$b_{l-m}=\frac{1}{m!}\left\{\frac{d^m}{ds}\left[\frac{N(s)}{D(s)}(s-p_i)^l\right]\right\}\Big|_{s=p_1},\ b_1=\frac{1}{(l-1)!}\left\{\frac{d^{l-1}}{ds}\left[\frac{N(s)}{D(s)}(s-p_i)^l\right]\right\}\Big|_{s=p_i}$$

Laplace Transform for Solving Diff. Eq.

Example: Solve the following differential equation

$$y^{(3)} + 3\ddot{y} + 3\dot{y} + y = 1, y(0) = \dot{y}(0) = \ddot{y}(0) = 0$$

$$s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0) + 3\left(s^2Y(s) - sy(0) - \dot{y}(0)\right)$$

$$+3\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s}$$

$$\left(s^3 + 3s^2 + 3s + 1\right)Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s\left(s^3 + 3s^2 + 3s + 1\right)} = \frac{1}{s(s+1)^3}$$

$$Y(s) = \frac{c_1}{s} + \frac{b_3}{(s+1)^3} + \frac{b_2}{(s+1)^2} + \frac{b_1}{s+1}$$

Laplace Transform for Solving Diff. Eq.

Determining coefficients:

$$c_1 = \frac{1}{s(s+1)^3} s \bigg|_{s=0} = 1$$

$$b_3 = [\frac{1}{s(s+1)^3}(s+1)^3]_{s=-1} = -1 \qquad b_1 = \frac{1}{2!}(2s^{-3}) \bigg|_{s=-1} = -1$$

$$b_2 = \left\{ \frac{d}{ds}[\frac{1}{s(s+1)^3}(s+1)^3] \right\}_{s=-1} = [\frac{d}{ds}(\frac{1}{s})]_{s=-1} = (-s^{-2})\Big|_{s=-1} = -1$$

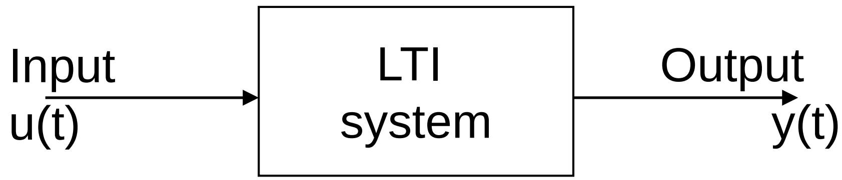
$$\therefore Y(s) = \frac{1}{s} - \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

Inverse Laplace transform:

$$y(t) = 1 - \frac{1}{2}t^2 e^{-t} - te^{-t} - e^{-t}$$

Try MATLAB functions:
laplace
ilaplace

Transfer Function



Consider a linear system described by differential equation

$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_0 y(t)= b_m u^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+bu^{(1)}(t)+b_0 u(t)$$

Assume all initial conditions are zero, we get the transfer function(TF) of the system as

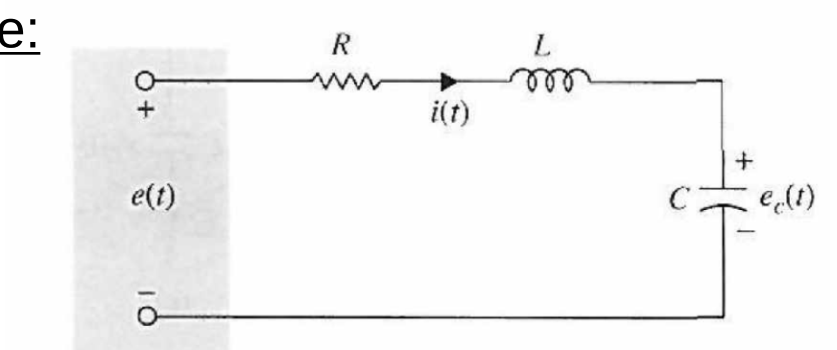
Try MATLAB functions:
tf(num,den)

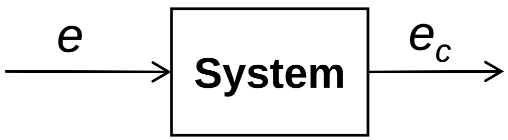
$$TF = G(s) = \left.\frac{\mathcal{L}\left[output\ \ y(t)\right]}{\mathcal{L}\left[input\ u(t)\right]}\right|_{zero\ \ initial\ \ condition}$$

$$= \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s + b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_1 s + a_0}$$

Transfer Function

Example:





$$e(t) = Ri(t) + L\frac{di}{dt} + e_c(t) \Leftarrow i(t) = C\frac{de_c(t)}{dt}$$

$$e(t) = RC\frac{de_c(t)}{dt} + LC\frac{d^2e_c(t)}{dt^2} + e_c(t)$$

$$\boxed{LC\ddot{e}_c + RC\dot{e}_c + e_c = e} \Leftarrow$$

i/o relationship
2nd order linear ordinary differential equation with constant coefficients

$$LCs^2E_c(s) + RCsE_c(s) + E_c(s) = E(s)$$

$$G(s) = \frac{E_c(s)}{E(s)} = \frac{1}{LCs^2 + RCs + 1} \Leftarrow$$

2nd order polynomial in denominator of TF

Transfer Function

Remarks:

- The transfer function is defined only for a LTI system
- All initial conditions of the system are set to zero
- The transfer function is independent of the input of the system
- The transfer function H(s) is the Laplace transform of the unit impulse response h(t)

$$h(t) = y(t)\Big|_{x(t)=\delta(t)} = \mathcal{L}^{-1}\left\{H(s)\cdot\mathcal{L}\left\{\delta(t)\right\}\right\}$$

$$= \mathcal{L}^{-1}\left\{H(s)\right\}$$

- What about Step Response (Output of the system when input is the unit step function)? How is it related to TF?

$$h_{step}(t) = y(t)\Big|_{x(t)=u(t)} = \mathcal{L}^{-1}\left\{H(s)\cdot\mathcal{L}\left\{u(t)\right\}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$$

Transfer Function

How poles and zeros relate to system response??

- Why we strive to obtain TF models?
- Why control engineers prefer to use TF model?
- How to use TF model to analyze and design control systems?

- we start from the relationship between the locations of zeros and poles of TF and the output responses of a system

Try MATLAB function:
tf2zp,tf
impulse
step
lsim

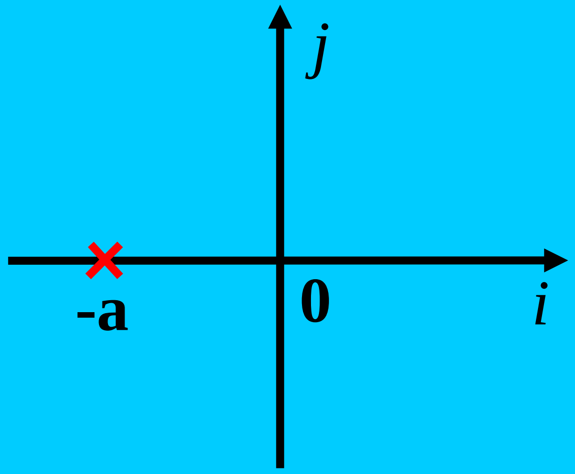
Transfer function

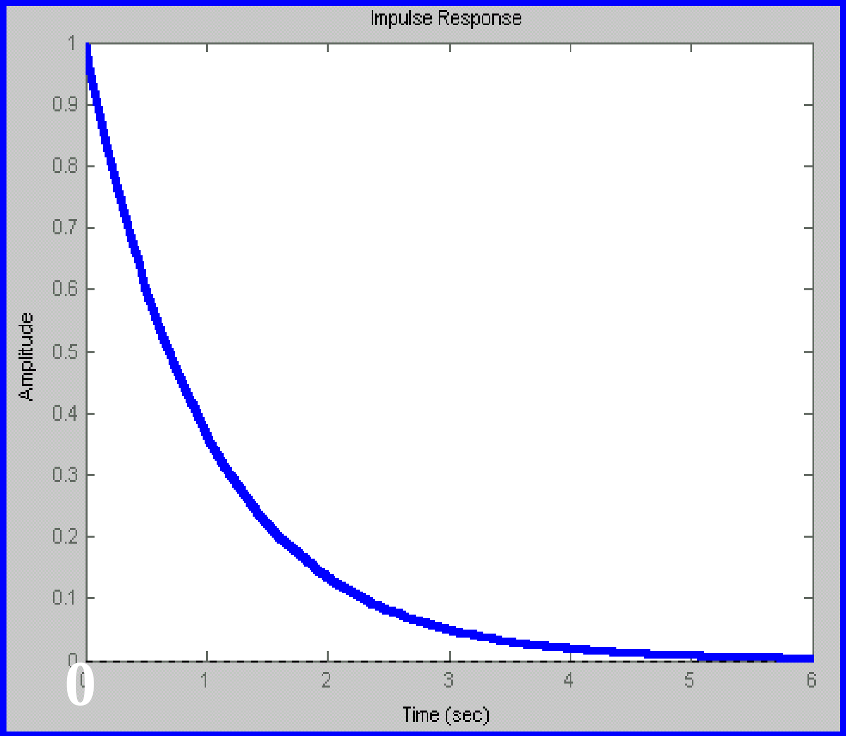
$$X(s) = \frac{A}{s+a}$$

Time-domain impulse response

$$x(t) = Ae^{-at}$$

Position of poles and zeros





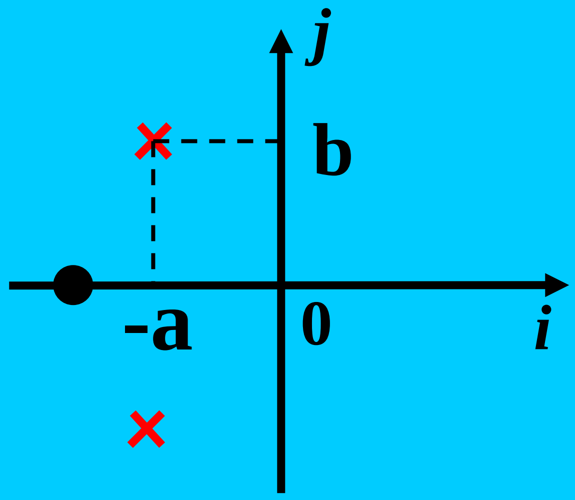
Transfer function

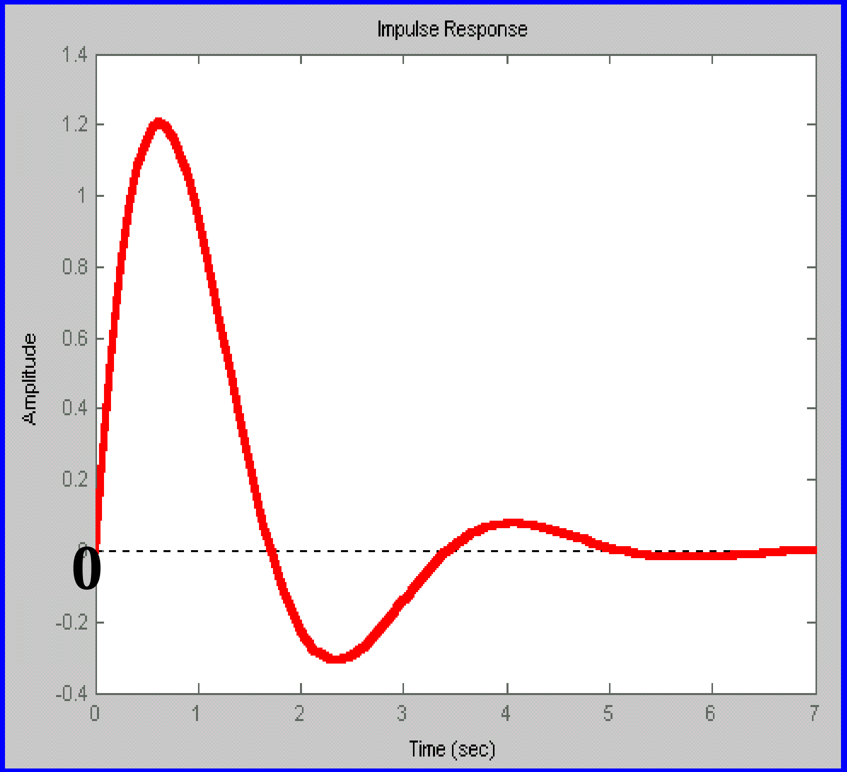
$$X(s) = \frac{A_1 s + B_1}{(s+a)^2 + b^2}$$

Time-domain impulse response

$$x(t) = Ae^{-at}\sin(bt + \phi)$$

Position of poles and zeros





Transfer function

$$X(s) = \frac{A_1 s + B_1}{s^2 + b^2}$$

Time-domain impulse response

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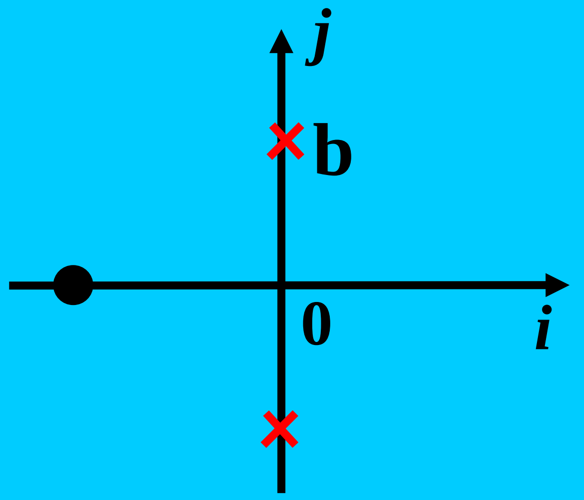
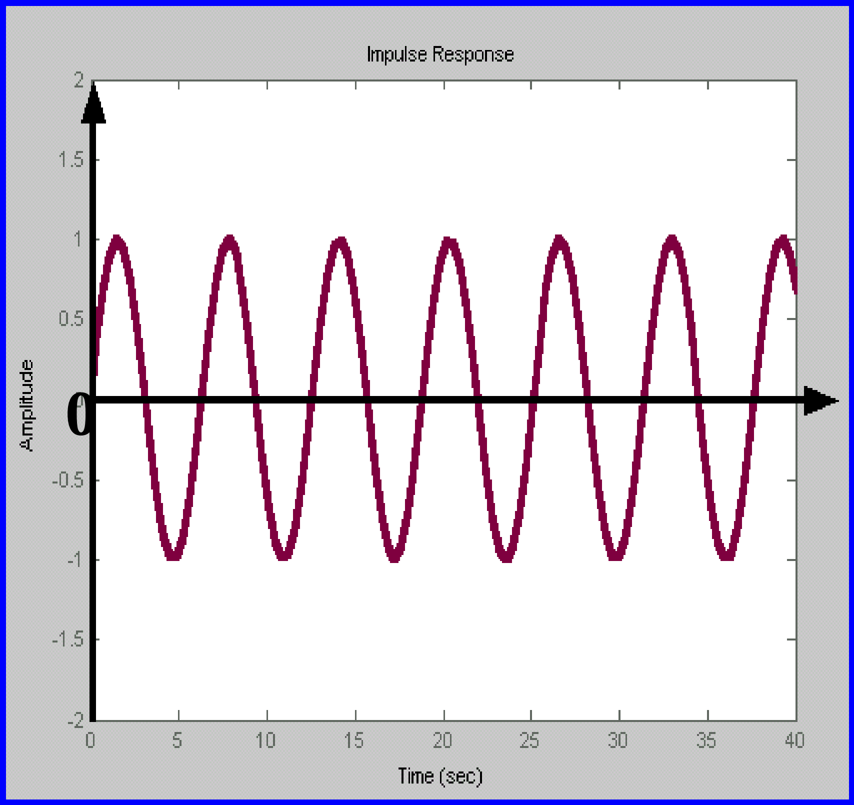
Position of poles and zeros





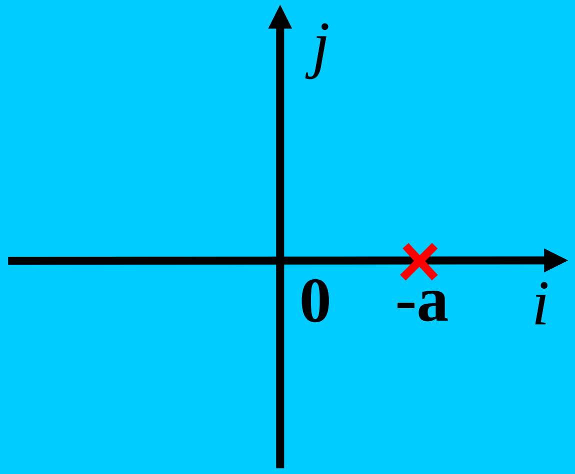
Transfer function

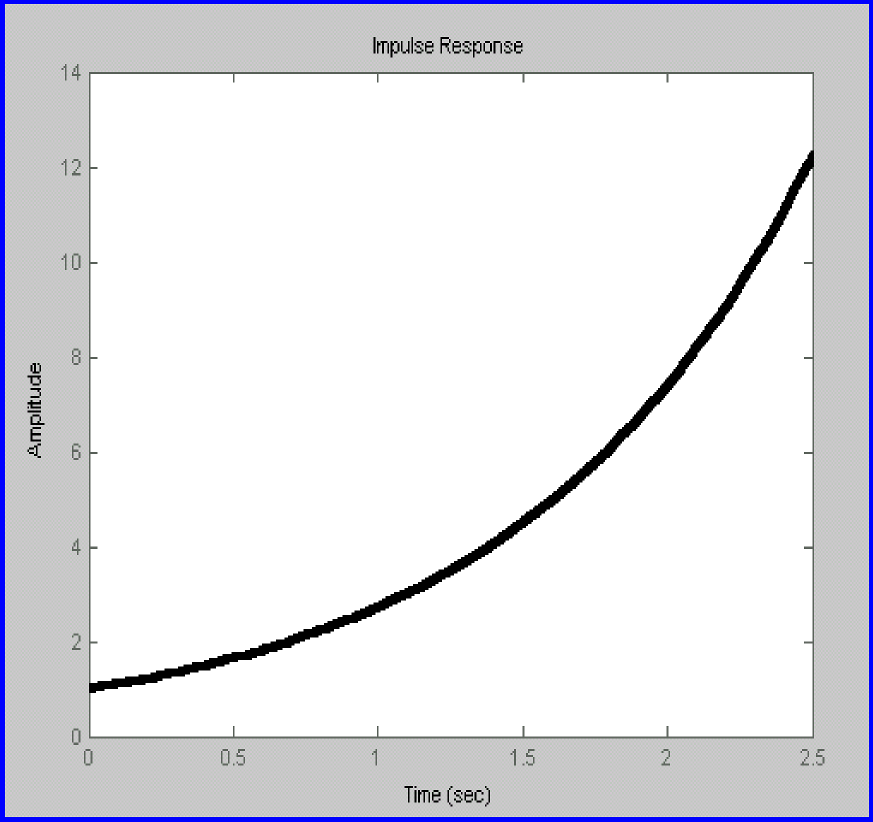
$$X(s) = \frac{A}{s-a}$$

Time-domain impulse response

$$x(t) = Ae^{at}$$

Position of poles and zeros





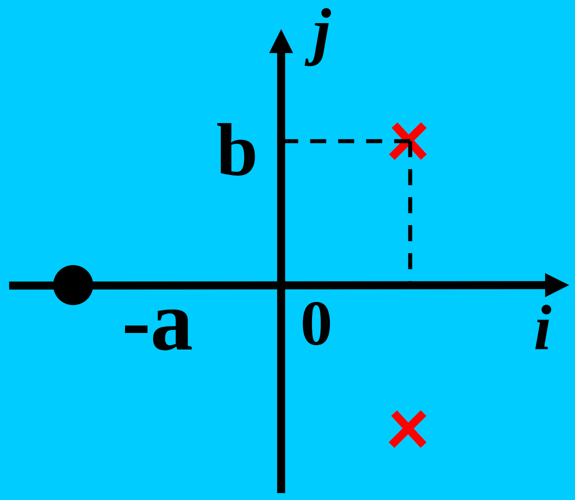
Transfer function:

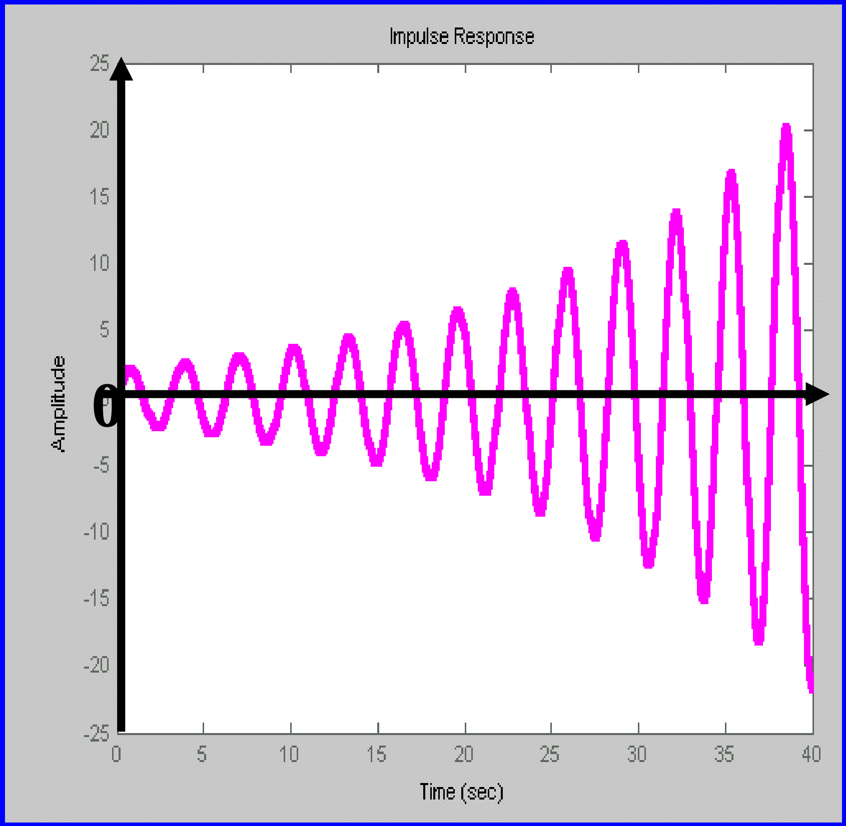
$$X(s) = \frac{A_1 s + B_1}{(s-a)^2 + b^2}$$

Time-domain dynamic response

$$x(t) = Ae^{at}\sin(bt + \phi)$$

Position of poles and zeros





Transfer Function

Characteristic equation

obtained by setting the denominator polynomial of the transfer function to zero

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

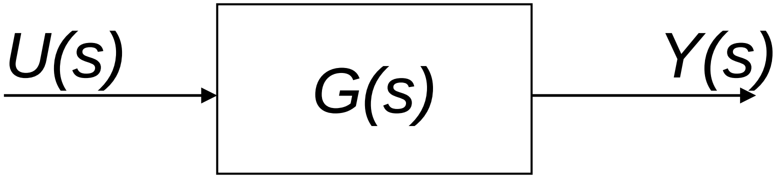
Note: stability of linear single-input, single-output systems is completely governed by the roots of the characteristics equation.

Block Diagram Representations

- The transfer function relationship

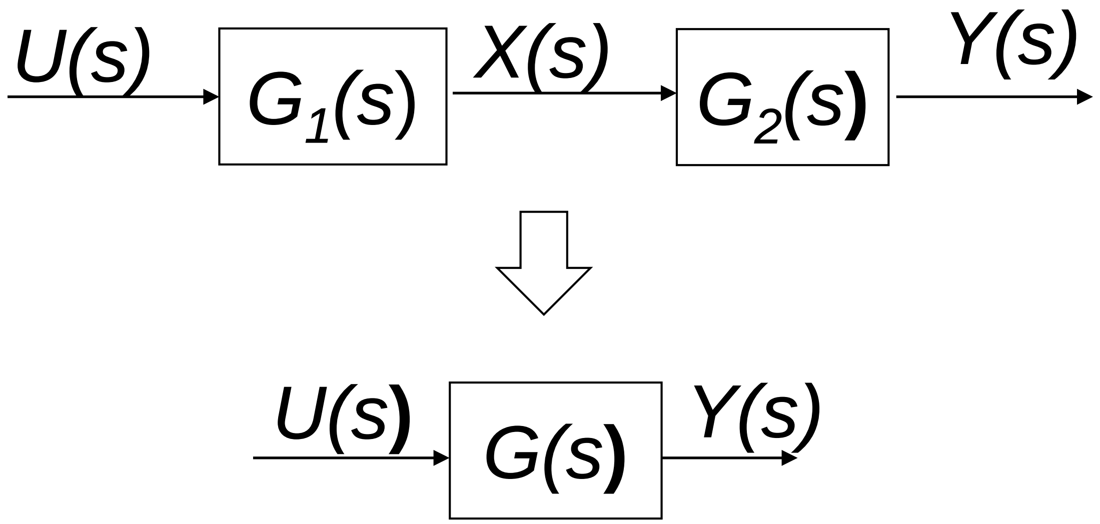
$$Y(s) = G(s)U(s)$$

can be graphically denoted through a block diagram.



Block Diagram Representations

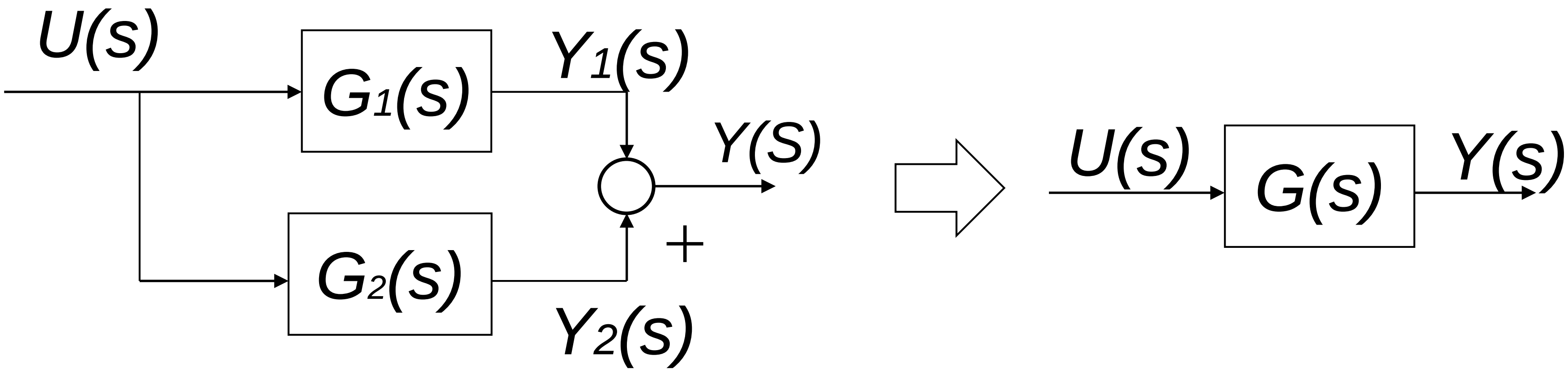
- Equivalent block diagram of two blocks in series (cascade)



$$G(s)=\frac{Y(s)}{U(s)}=\frac{X(s)}{U(s)}\cdot\frac{Y(s)}{X(s)}=G_1(s)\cdot G_2(s)$$

Block Diagram Representations

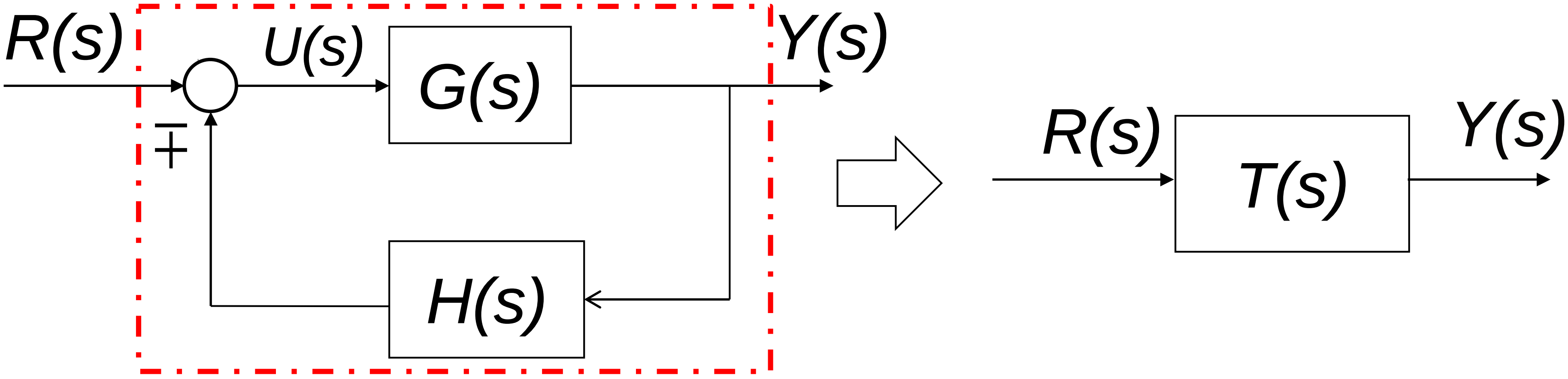
- Equivalent block diagram of two blocks in parallel



$$G(s) = \frac{Y(s)}{U(s)} = \frac{Y_1(s) + Y_2(s)}{U(s)} = G_1(s) + G_2(s)$$

Block Diagram Representations

- Equivalent block diagram of a feedback system



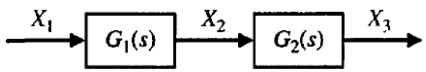
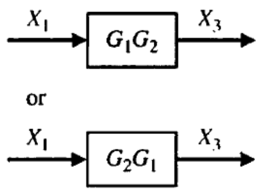
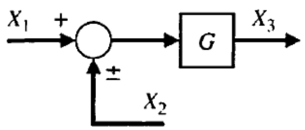
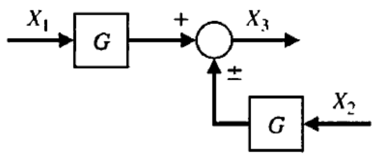
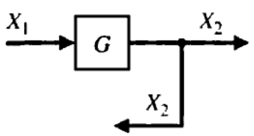
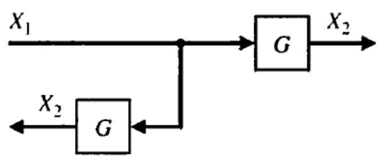
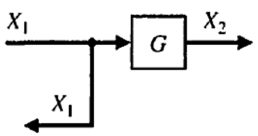
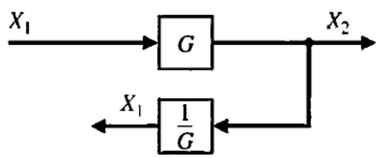
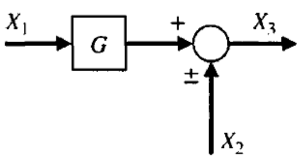
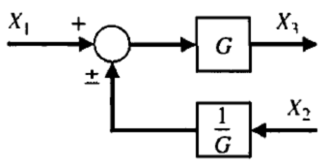
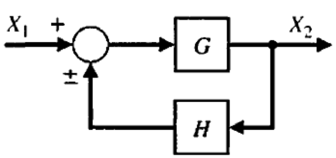
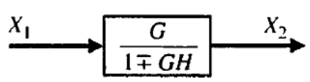
$$\begin{cases} Y(s) = U(s)G(s) \\ U(s) = R(s) - Y(s)H(s) \end{cases} \Rightarrow Y(s) = \left[R(s) - Y(s)H(s)\right]G(s)$$

$$\Downarrow$$

$$T(s) = \frac{G(s)}{1 \pm G(s)H(s)} = \frac{\text{gain of forward path}}{\text{1-loop gain}}$$

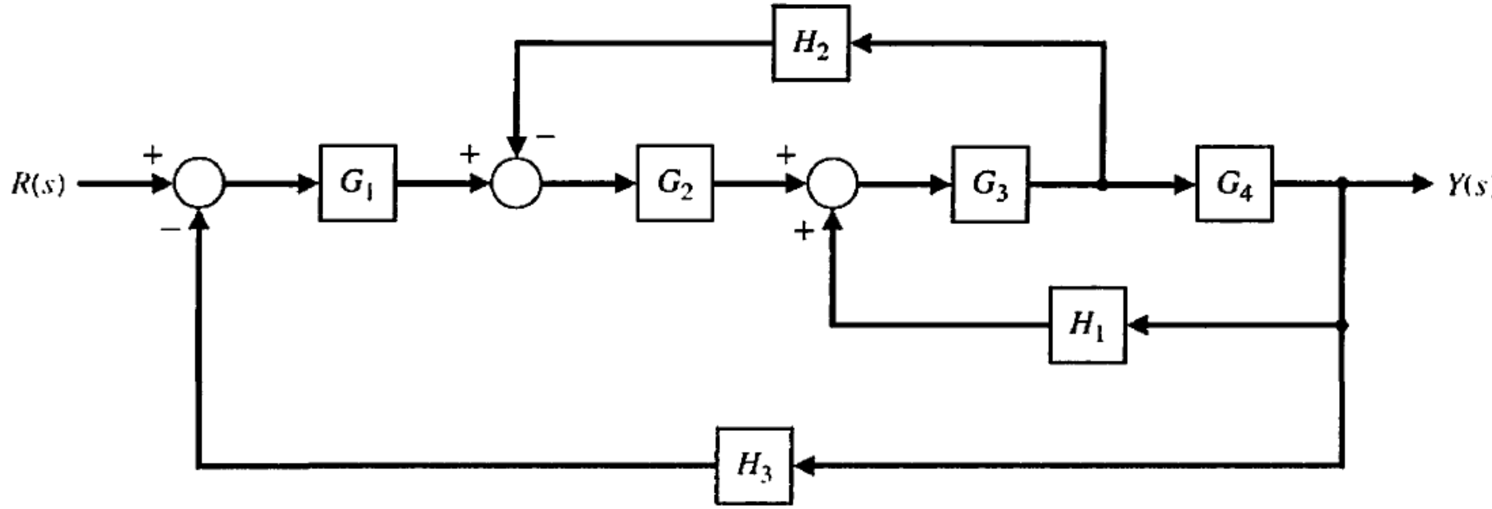
Block Diagram Representations

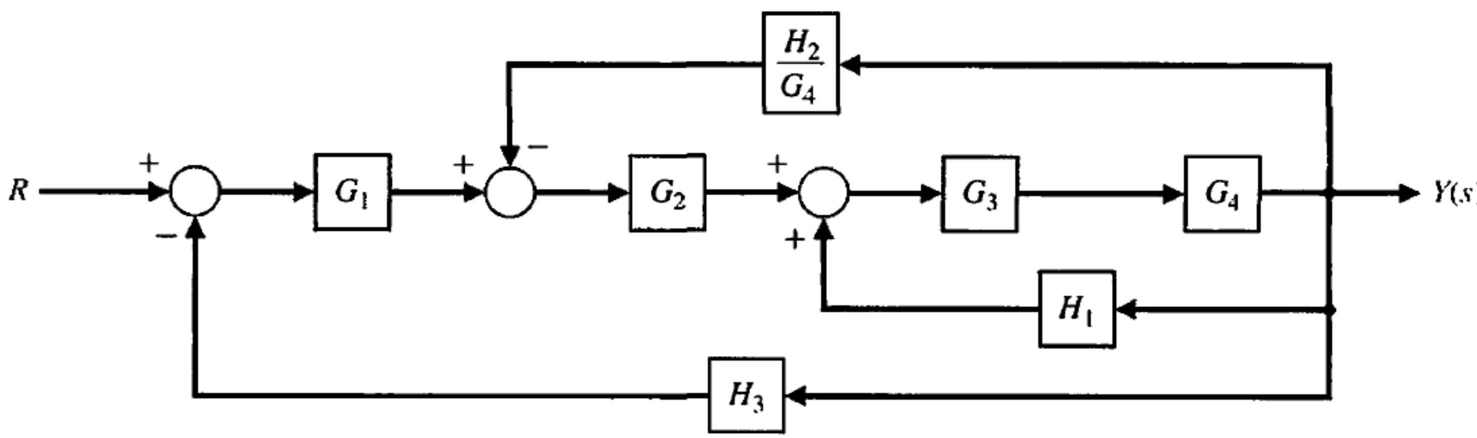
- <u>Summary</u>

Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

Block Diagram Representations

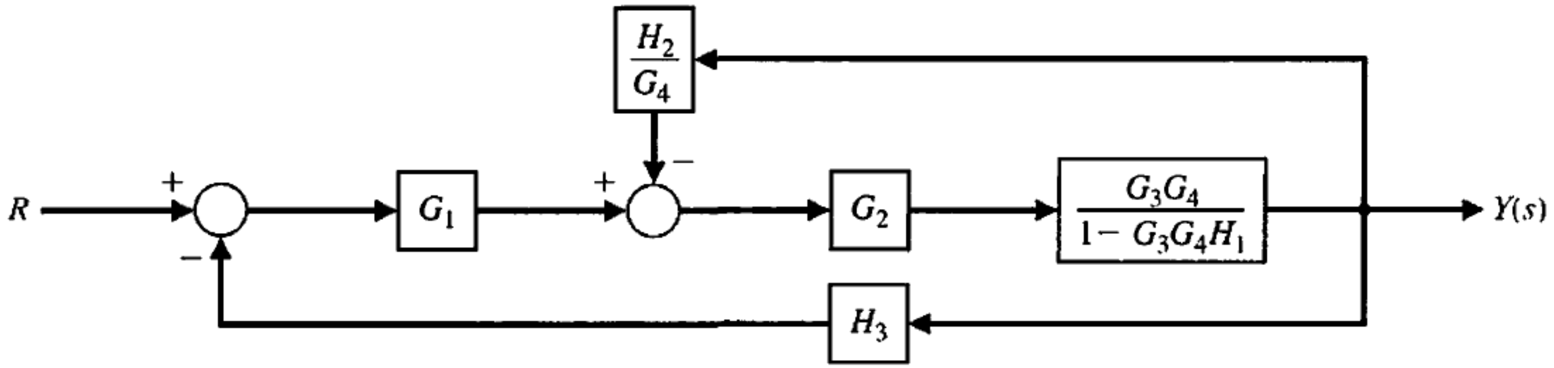
- Example

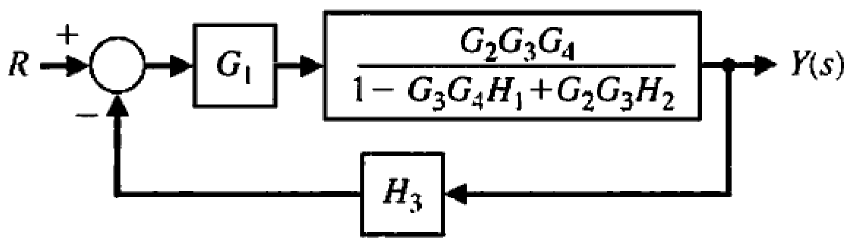


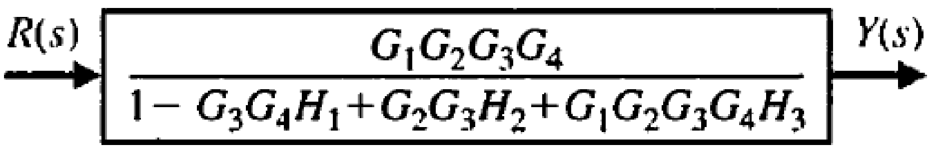


Block Diagram Representations

- <u>Example (cont.)</u>







$$\frac{G_1G_2G_3G_4}{1-G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4H_3}$$